K3 surfaces, assignment 8: integrability of almost complex structures

Exercise 8.1. Let (M, I) be an almost complex manifold, $\dim_{\mathbb{C}} M = n$, $U \subset M$ a dense, opens subset, and $\Omega \in \Lambda^{n,0}(U)$ a non-degenerate $(n, 0)$-form. Assume that $d\Omega = 0$. Prove that the almost complex structure I is integrable.

Exercise 8.2. Let (M, I) be an almost complex manifold, and W the Weil operator, acting on (p, q)-forms as $W(\eta) = \sqrt{-1}(p - q)\eta$. Prove that I is integrable if and only if $I^{-1} dI - [W, d] = 0$.

Definition 8.1. Let G be a real Lie group, and $\mathfrak{g} = T_e G$ its Lie algebra. **The left action** of G on itself is a map $L_g(x) = gx$, defined for any $g \in G$. An almost complex structure is called **left invariant** if it is preserved by L_g for all $g \in G$.

Exercise 8.3. Let G be a real Lie group, and $\mathfrak{g} = T_e G$ its Lie algebra, and $\mathfrak{g}_{\mathbb{C}} := \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$ its complexification.

a. Prove that any complex structure operator on $\mathfrak{g} = T_e G$ is uniquely extended to a left-invariant almost complex structure on G.

b. Let $I \in \operatorname{End} T_e G$ be a complex structure, and $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g}^{1,0} \oplus \mathfrak{g}^{0,1}$ the Hodge decomposition of its complexification. Prove that I defines a formally integrable left-invariant complex structure on G if and only if $\mathfrak{g}^{1,0} \subset \mathfrak{g}_{\mathbb{C}}$ is a Lie subalgebra.

Exercise 8.4. Let (M, I) be an almost complex manifold. Assume that the Nijenhuis tensor $\Lambda^2(T^{1,0}M) \longrightarrow T^{0,1}(M)$ (that is, the Frobenius for of the distribution $T^{1,0}M$) is surjective. Prove that (M, I) admits no local holomorphic functions.

Exercise 8.5. Let $B \subset TM$ be a sub-bundle.

a. Prove that there exists a connection $\nabla : TM \longrightarrow TM \otimes \Lambda^1(M)$ such that $\nabla(B) \subset B \otimes \Lambda^1(M)$.

b. Suppose that ∇ is torsion-free. Prove that B is an integrable distribution, that is, $[B, B] \subset B$.

c. Assume that $[B, B] \subset B$. Prove that then ∇ can be chosen torsion-free.