K3 surfaces, assignment 9: Moser isotopy lemma

Exercise 9.1. Let M be a compact manifold, and V_0, V_1 two smooth volume forms which satisfy $\int_M V_0 = \int_M V_1$. Prove that there exists a diffeomorphism which satisfies $\Phi^* V_0 = V_1$.

Exercise 9.2. Let (M, I) be an almost complex manifold, and ω_0, ω_1 cohomologous symplectic forms which satisfy $\omega_i(x, Ix) > 0$ for any non-zero tangent vector x (such forms are called **taming**).

a. Prove that there exists a diffeomorphism Φ which satisfies $\Phi^*\omega_0 = \omega_1$.

b. Prove that $|x|_i^2 := \omega_i(x, Ix)$ is a Hermitian metric on M. Prove that a diffeomorphism that satisfies $\Phi^*\omega_0 = \omega_1$ defines an isometry

$$(M, |x|_1^2) \longrightarrow (M, |x|_0^2)$$

if Φ is compatible with I, that is, satisfies $d\Phi(Ix) = I(d\Phi(x))$.

c. Find examples of ω_0, ω_1 taming (M, I) such that a diffeomorphism compatible with I and satisfying $\Phi^*\omega_0 = \omega_1$ does not exist.

Exercise 9.3. Let $\omega_i \in \Lambda^2(M)$ be a sequence of symplectic forms on a compact manifold M converging to a symplectic form ω in C^0-topology. Assume that all ω_i are homologous. Prove that almost all (M, ω_i) are symplectomorphic.

Exercise 9.4. Let (M, V) be a connected manifold equipped with a volume form. Prove that the group of volume-preserving diffeomorphisms acts on M transitively.

Exercise 9.5. Prove that the group of symplectomorphisms of (M, ω) acts transitively on M, for any connected symplectic manifold (M, ω).

Exercise 9.6 (*). Let Ψ be a diffeomorphism of a connected symplectic manifold (M, ω) which satisfies $\Psi^*\omega = 2\omega$. Prove that ω is exact, or find a counterexample.