K3 surfaces, exam

Rules: The final grade is determined by the score. The mark is C for score 1, B for 2, A for 3, A+ for the higher score.

Exercise 1.1 (1 point). Let M be a compact complex surface. Prove that all holomorphic differential forms on M are closed.

Definition 1.1. An almost hypercomplex structure on a manifold M is a triple almost complex structures (I, J, K) satisfying the quaternionic relations. It is called **hypercomplex** if I, J, K are integrable. **An almost hypercomplex Hermitian structure** on M is an almost complex structure (I, J, K) and a Riemannian metric h which is invariant under the action of I, J, K. It is called **hyperkähler** if I, J, K are complex, and h is Kähler with respect to I, J, K.

Exercise 1.2 (1 point). Prove that any almost hypercomplex manifold (M, I, J, K) satisfies $c_1(M, I) = 0$.

Exercise 1.3 (1 point). Let (M, I, J, K) be an almost hypercomplex Hermitian manifold, and
$\omega_I, \omega_J, \omega_K$ its fundamental forms. Suppose that these forms are closed. Prove that (M, I, J, K) is hyperkähler.

Exercise 1.4 (2 points). Let (M, I, J, K) be a hypercomplex Hermitian manifold, and $\omega_I, \omega_J, \omega_K$ its fundamental forms. Suppose that ω_I is closed. Prove that ω_J, ω_K are closed, or find a counterexample.

Exercise 1.5 (3 points). Let (M, I, J, K) be an almost hypercomplex Hermitian manifold, and $\omega_I, \omega_J, \omega_K$ its fundamental forms. Suppose that ω_I, ω_J are closed. Prove that ω_K is closed, or find a counterexample.

Exercise 1.6 (3 points). Let $\omega_1, \omega_2, \omega_3$ be a triple of 2-forms on a 4-manifold M such that any non-zero linear combination of ω_i is non-degenerate. Prove that there exists an almost hypercomplex Hermitian structure with fundamental forms $\omega_I, \omega_J, \omega_K$ such that the 3-dimensional sub-bundles of $\Lambda^2(M)$ spanned by $\omega_I, \omega_J, \omega_K$ and $\omega_1, \omega_2, \omega_3$ coincide.

Exercise 1.7 (2 points). Let $Z \subset (M, \Omega)$ be a submanifold in a holomorphically symplectic manifold. Assume that Z is Lagrangian with respect to the symplectic forms $Re\,\Omega$ and $Im\,\Omega$. Prove that Z is complex analytic.

Definition 1.2. A holomorphic Lagrangian fibration on a holomorphically symplectic manifold (M, Ω) is a holomorphic submersion $\pi : \ M \longrightarrow X$ such that the fibers of π are holomorphic Lagrangian with respect to Ω.

Exercise 1.8 (2 point). Let $\pi : \ M \longrightarrow X$ be a holomorphic Lagrangian fibration, and $\sigma : \ X \longrightarrow M$ a smooth section. Prove that $\sigma^*(\Omega)$ has Hodge type $(2, 0) + (1, 1)$.

Exercise 1.9 (2 points). Let (M, I, J, K) be a hypercomplex manifold, $d_I := IdI^{-1}, d_J := JdJ^{-1}, d_K := KdK^{-1}$, and $D := dd_I d_J d_K : \ \Lambda^*(M) \longrightarrow \Lambda^{*+4}(M)$. Prove that D is independent from the choice of a basis I, J, K in quaternions.