# K3 surfaces, home assignment 1: Riemann-Roch formula in dimension 1

**Rules:** This is a class assignment for this week. Please bring your solutions (written) next Monday. We will have a class discussion the Wednesday after.

**Remark 1.1.** The Riemann–Roch formula for the curve is $\chi(F) = \deg(F) + \chi(\mathcal{O}_C)\operatorname{rk} F$. Here we deduce this formula, together with $\deg(B) = \int_C c_1(B)$ for any vector bundle $B$ on $C$. However, both the degree and $c_1$ are defined in such a way that the Riemann–Roch formula becomes a part of their definition.

**Definition 1.1.** A **principal ideal** in a ring $R$ is an ideal $xR$ generated by an element $x \in R$. A **principal ideal ring** is a ring where all ideals are principal.

**Exercise 1.1.** Prove that the ring $\mathcal{O}_1$ of germs of holomorphic functions on $\mathbb{C}$ is a principal ideal ring.

**Exercise 1.2.** ("Invariant factors theorem"). Let $R$ be a principal ideal ring. Prove that any finitely–generated $R$-module is a direct sum of cyclic $R$-modules. Use this result to deduce the Jordan normal form theorem, and to classify the finitely–generated abelian groups.

**Definition 1.2.** A **coherent sheaf** on a complex manifold $M$ is a sheaf of modules over the sheaf $\mathcal{O}_M$ of holomorphic functions on $M$, which is locally finitely generated and locally finitely presented (that is, the sheaf of relations between its local generators is also locally finitely generated).

**Exercise 1.3.** Let $C$ be a complex curve, and $x \in C$ a smooth point. Prove that any coherent sheaf on $C$ supported in $x$ is isomorphic to $\bigoplus_{i=1}^k \mathcal{O}_C/\mathfrak{m}^{d_i}$, where $\mathfrak{m}$ is the maximal ideal of $x$, and $d_1, ..., d_k$ is a collection of positive integers.

**Hint.** Use the previous exercise.

**Remark 1.2.** In the following exercises, you can freely assume that any compact complex curve admits a line bundle $L$ with the following property. For any holomorphic vector bundle $B$, there exists $n \gg 0$ such that the tensor power $B \otimes L^{\otimes n}$ is globally generated. This result is deduced from Kodaira-Nakano vanishing theorem. Also, we assume that $H^0(F)$ and $H^1(F)$ is finite-dimensional for any coherent sheaf $F$, and $H^i(F) = 0$ for all $i > 1$.

**Exercise 1.4.** Let $C$ be a complex curve, and $V$ an abelian group, freely generated by isomorphism classes of coherent sheaves on $C$. **The Grothendieck K-group** $K_0(C)$ is the quotient of $V$ by its subgroup generated by relations $[F_1] + [F_3] = [F_2]$ for all exact sequences of coherent sheaves $0 \longrightarrow F_1 \longrightarrow F_2 \longrightarrow F_3 \longrightarrow 0$.

a. Let $L$ be a line bundle, and $0 \longrightarrow \mathcal{O}_C \longrightarrow L \longrightarrow R \longrightarrow 0$ be an exact sequence associated with a section $l \in H^0(C, L)$. Prove that $[L] - [\mathcal{O}_C] = \sum a_i[x_i]$, where $a_i \in \mathbb{Z}^{>0}$, $[x_i]$ are classes of skyscraper sheaves $\mathcal{O}_C/\mathfrak{m}_{x_i}$, and $\mathfrak{m}_{x_i}$ is the maximal ideal of a point $x_i$.

b. Prove that $K_0(C)$ is generated by $\mathcal{O}_C$ and the classes of skyscraper sheaves $\mathcal{O}_C/\mathfrak{m}_x$.

**Exercise 1.5.** Let $C$ be a compact complex curve, and $F$ a coherent sheaf on $C$. We define **the Euler characteristic of** $F$ as $\chi(F) := \dim H^0(C, F) - \dim H^1(C, F)$. Prove that $\chi$ defines a group homomorphism $K_0(C) \longrightarrow \mathbb{Z}$.

**Exercise 1.6.** Consider line bundles on a compact complex curve $C$.

a. Let $L$ be a line bundle, admitting a holomorphic section, and

$$0 \longrightarrow \mathcal{O}_C \longrightarrow L \longrightarrow R \longrightarrow 0$$

the corresponding exact sequence. Define the **degree** $\deg L$ as $\chi(L) - \chi(\mathcal{O}_C)$. Prove that $\deg(L) = \dim H^0(C, R)$, where $R$ is defined above.

b. Prove that the degree is multiplicative, $\deg(L_1 \otimes L_2) = \deg(L_1) + \deg(L_2)$.

**Exercise 1.7.** Let $X$ be a complex manifold, and $\operatorname{Pic}(X)$ the set of equivalence classes of vector bundles, equipped with the multiplicative structure induced by the tensor product. The group $\operatorname{Pic}(X)$ is called **the Picard group of $X$.**

a. Prove that the cohomology group $H^1(X, \mathcal{O}_X^*)$ is naturally identified with $\operatorname{Pic}(X)$.

b. Consider **the exponential exact sequence** $0 \longrightarrow \mathbb{Z}_X \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_X^* \longrightarrow 0$, where $\mathbb{Z}_X$ denotes the constant sheaf. The corresponding long exact sequence

$$\longrightarrow H^1(X, \mathcal{O}_X) \longrightarrow H^1(X, \mathcal{O}_X^*) \longrightarrow H^2(X, \mathbb{Z}) \longrightarrow$$

takes a line bundle $[L] \in \operatorname{Pic}(X) = H^1(X, \mathcal{O}_X^*)$ to an element $c_1(L) \in H^2(X, \mathbb{Z})$ called **the first Chern class of $L$**. Prove that a non-trivial bundle $L$ with $c_1(L) = 0$ on a compact complex curve has no holomorphic sections.

c. (*) Construct a non-trivial bundle $L$ on a compact complex manifold such that $c_1(L) = 0$, but $H^0(L) \neq 0$.

**Exercise 1.8.** Let $C$ be a complex curve and $F$ a coherent sheaf on $C$.

a. Prove that the restriction of $F$ to a certain open set $U \subset C$ is isomorphic to a vector bundle. Prove that the rank of this vector bundle is independent on the choice of $U$ when $C$ is irreducible. This number is called **the rank** of $F$.

b. Denote by $[x] \in H^2(C, \mathbb{Z})$ the fundamental class of a point, that is, the generator of the group $H^2(C, \mathbb{Z}) = \mathbb{Z}$. Define **the degree** of a coherent sheaf $F$ as $\deg_C(F) := \chi(F) - \mathsf{rk}(F)$, and let $c_1(F) := \deg_C F \cdot [x]$ be **the first Chern class of $F$**. Prove that the first Chern class defines a group homomorphism $c_1 : K_0(C) \longrightarrow H^2(C, \mathbb{Z})$.

c. Prove that this definition is compatible with the definition of $c_1(L)$ for line bundles given above.

d. Prove that $c_1$ satisfies **the Whitney formula**: for any two vector bundles $B_1, B_2$ on a curve, $c_1(B_1 \oplus B_2) = c_1(B_1) + c_1(B_2)$.

e. Let $B_1, B_2$ be vector bundles on $C$. Prove that $c_1(B_1 \otimes B_2) = \mathsf{rk}\, B_1 \cdot c_1(B_2) + \mathsf{rk}\, B_2 \cdot c_1(B_1)$.