K3 surfaces, home assignment 2: spectral sequences

Rules: This is a class assignment for this week. Please bring your solutions (written) next Monday. We will have a class discussion the Wednesday after.

2.1 The monodromy of Gauss-Manin local system

Definition 2.1. Let $\pi : E \longrightarrow B$ be a locally trivial fibration with fiber F. The family of cohomology of a fibers of π is locally trivial, but it might have **the monodromy**. In other words, the group $\pi_1(B)$ naturally acts on the algebra $H^*(F)$ by automorphisms. To obtain this action, take a loop in B and trivialize the family π along small intervals in this loop; this gives an identification of $H^*(F)$ with itself, which might be non-trivial.

Exercise 2.1. Let $\phi^* : \mathbb{Z} \longrightarrow \operatorname{Aut}(H^*(F))$ be an automorphism induced by a homeomorphism $\phi : F \longrightarrow F$. Construct a locally trivial family over a circle with monodromy in cohomology induced by ϕ^*.

Exercise 2.2. Let K be a Klein bottle. Consider the standard fibration $\pi : K \longrightarrow S^1$ with the fiber $F = S^1$. Find the monodromy action on $H^1(F)$.

Exercise 2.3. Let $\phi : M \longrightarrow S$ be a smooth holomorphic submersion with the base a complex curve and fiber F an elliptic curve. Prove that the monodromy acts trivially on $H^2(F)$. Prove that the monodromy acts trivially on $H^1(F)$ or find a counterexample.

Remark 2.1. In the sequel I would describe the Leray-Serre spectral sequence for families with trivial monodromy. It is not hard to extend this description to families with non-trivial monodromy, using the language of local systems.

2.2 Leray-Serre spectral sequence

I am going to describe the spectral sequence as a computational recipy, with no background given. Throughout this section, $\pi : E \longrightarrow B$ is a locally trivial fibration (or, more generally, a Serre's fibration), with fiber F. We assume that the monodromy action of $\pi_1(B)$ on $H^*(F)$ is trivial. We also assume that the cohomology are taken with coefficients in a field of $\mathsf{char} = 0$.

Definition 2.2. Let $H^{\geqslant i}(B)H^*(E)$ denote the subgroup in $H^*(E)$ multiplicatively generated by the pullbacks of $H^{\geqslant i}(B)$. **The Leray-Serre filtration** on $H^*(E)$ is $H^*(E) \supset H^{\geqslant 1}(B)H^*(E) \supset H^{\geqslant 2}(B)H^*(E) \supset ...$ This filtration is clearly multiplicative. The associated graded ring

$$\bigoplus_{p,q} E_\infty^{p,q}, \text{ where } E_\infty^{p,q} := \frac{H^p(B)H^q(E)}{H^{\geqslant p+1}(B)H^*(E)}$$

is isomorphic to $H^*(E)$ as a vector space (if the base coefficients are a field); however, the ring structure is often different. **The Leray-Serre spectral sequence** is a collection of spaces $E_i^{p,q}$, with $i \geqslant 2$ and differentials $d_i : E_i^{p,q} \longrightarrow E_i^{p+i,q-i+1}$, $d_i^2 = 0$, with the following properties.

- $E_2^{p,q} = H^p(B) \otimes H^q(F)$, considered as a ring. For each i, the space $\oplus_{p,q} E_i^{p,q}$ is a graded commutative ring.

- The differentials d_i acting on $\oplus_{p,q} E_i^{p,q}$ satisfy the Leibnitz rule, and $E_{i+1}^{p,q}$ is equal to the cohomology algebra of $(\oplus_{p,q} E_i^{p,q}, d_i)$.
- $E_\infty^{p,q} = \lim_i E_i^{p,q}$ (this is expressed by saying "the spectral sequence converges to $E_\infty^{p,q}$")

The ring $\bigoplus_{p,q} E_k^{p,q}$ is called **the k-th page of the spectral sequence**. If $d_i = 0$ for $i \geqslant k$, we say that the spectral sequence **degenerates at the page k.**

Exercise 2.4. Let $\pi: E \longrightarrow B$ be a fibration with the fiber a torus. Assume that $d_2 = 0$. Prove that all differentials d_i vanish.

Exercise 2.5. Let $\pi: E \longrightarrow B$ be a fibration with the fiber a torus. Assume that the pullback map $\pi^* H^2(B) \longrightarrow H^2(E)$ is injective. Prove that all differentials d_i vanish.

Hint. Use the previous exercise.

Exercise 2.6. Let $\pi: E \longrightarrow B$ be a fibration with the fiber a complex projective space. Assume that $d_2 = 0$ and $d_3 = 0$. Prove that all differentials d_i vanish.

Exercise 2.7. Let $\tau: F \longrightarrow E$ be the standard embedding map. Prove that the sequence

$$0 \longrightarrow H^1(B) \xrightarrow{\pi^*} H^1(E) \xrightarrow{\tau^*} H^1(F) \xrightarrow{d_2} H^2(B) \xrightarrow{\pi^*} H^2(E)$$

is exact.

Exercise 2.8. Let $F = S^k$, that is, $\pi: E \longrightarrow B$ is a sphere bundle. Prove that all differentials except d_{k+1} vanish. Construct **the Gysin exact sequence**

$$... \longrightarrow H^p(B) \longrightarrow H^{p+k+1}(B) \xrightarrow{\pi^*} H^{p+k+1}(E) \longrightarrow H^{p+1}(B) \longrightarrow ...$$

Hint. Note that $E_2^{p,q} = H^p(B)$ for $q = 0, k$ and $E_2^{p,q} = 0$ otherwise.

Exercise 2.9. Let $\pi: E \longrightarrow B$ be a fibration with E contractible. Assume that F a $(2q-1)$-dimensional sphere. Prove that $H^*(B)$ is a free polynomial algebra with one generator in dimension $2q$.

Hint. Use the Gysin sequence and apply multiplicativity of the spectral sequence.

Exercise 2.10. Let $\pi: E \longrightarrow B$ be a fibration with $B = S^k$. Prove that all differentials except d_k vanish. Construct an exact sequence

$$... \longrightarrow H^{p+k}(F) \xrightarrow{\tilde{d}_k} H^p(F) \xrightarrow{\mu} H^{p+k}(E) \longrightarrow H^{p+k+1}(F) \longrightarrow ...$$

where μ is multiplication by $\pi^* \operatorname{Vol}_{S^d}$, and $\tilde{d}_k$ is equal to d_k after the identification $H^p(F) = H^k(S^k) \otimes H^p(F) = E_2^{k,p}$.

Exercise 2.11. Let $\pi: E \longrightarrow B$ be a fibration with the fiber F an odd-dimensional sphere. Assume that the rings $H^*(B)$ and $H^*(E)$ are freely generated by generators in odd degrees. Prove that all d_i vanish.

Exercise 2.12. Let $\pi: E \longrightarrow B$ be a fibration with E contractible, and $H^*(B)$ and $H^*(F)$ free graded algebras, with W^* the space of generators of $H^*(F)$. Prove that $d_i\Big|_{W^{i-1}} \longrightarrow H^i(B)$ is injective, and $\bigoplus_i d_i(W^{i-1})$ freely generates $H^*(B)$.