

K3 surfaces, home assignment 3: the splitting principle

Rules: This is a class assignment for this week. Please bring your solutions (written) next Monday. We will have a class discussion the Wednesday after.

Exercise 3.1. Let $\mathbb{C}P^\infty = \mathcal{G}\tau(1, \infty)$ be the union $\bigcup_i \mathbb{C}P^i$ where all maps $\mathbb{C}P^1 \hookrightarrow \mathbb{C}P^2 \hookrightarrow \mathbb{C}P^3 \hookrightarrow \dots$ are hyperplane embeddings. Prove that there exists a principal $U(1)$ -bundle over $\mathbb{C}P^\infty$ with contractible total space. Prove that the cohomology of $\mathbb{C}P^\infty$ is a polynomial algebra with one generator in $H^2(\mathbb{C}P^\infty)$.

Definition 3.1. The **fundamental bundle** on $\mathbb{C}P^\infty = BU(1)$ is B_{fun} , isomorphic to $\mathcal{O}(1)$ on each $\mathbb{C}P^n \subset \mathbb{C}P^\infty$.

Exercise 3.2. Let X be a compact CW-space. Prove that any line bundle on X is isomorphic to $\phi^*(B_{\text{fun}})$ for some continuous map $\phi: X \rightarrow BU(1)$.

Exercise 3.3. Let B_{fun} be the **fundamental vector bundle** on $\mathcal{G}\tau(n)$, which has fiber W at any point of $\mathcal{G}\tau(n)$ corresponding to a subspace $W \subset \mathbb{C}^\infty$. Let X be a CW-space. Prove that any vector bundle on X is isomorphic to $\phi^*(B_{\text{fun}})$ for some continuous map $\phi_B: X \rightarrow \mathcal{G}\tau(n)$.

Exercise 3.4. Let $\Phi: (BU(1))^n \rightarrow \mathcal{G}\tau(n)$ be a morphism such that the pullback of the fundamental bundle is the direct sum of n line bundles, obtained by lifting $\mathcal{O}(1)$ from each factor $BU(1)$. A complex vector bundle is called **split** if it obtained as a direct sum of complex line bundles. Prove that a vector bundle B on X is split if and only if $\phi_B: X \rightarrow \mathcal{G}\tau(n)$ is factorized through Φ .

Exercise 3.5. a. Let $\mathfrak{F}(V)$ be the space of all orthogonal bases in $V = \mathbb{C}^{n+1}$ up to independent rescaling of each of the vectors (“the flag space”; we will denote it \mathfrak{F}). Denote by \mathfrak{S} the smooth, locally trivial bundle over $\mathcal{G}\tau(n)$, with the fiber the flag space $\mathfrak{F}(V)$ in each subspace $V \in \mathcal{G}\tau(n)$. Prove that the pullback of the fundamental bundle B_{fun} to \mathfrak{S} is split.

b. Prove that the induced map $H^*(\mathcal{G}\tau(n), \mathbb{Q}) \rightarrow H^*(\mathfrak{S}, \mathbb{Q})$ is injective.

c. Deduce that $H^*(\mathfrak{S}, \mathbb{Q})$ as $H^*(\mathcal{G}\tau(n), \mathbb{Q})$ -module is isomorphic to $H^*(\mathcal{G}\tau(n), \mathbb{Q}) \otimes H^*(\mathfrak{F})$.

Hint. Construct a cell decomposition of \mathfrak{F} with all cells even-dimensional and use the Leray-Serre spectral sequence, or use the next exercise instead.

Exercise 3.6. Let $E \rightarrow B$ be a locally trivial fibration with fiber F . Assume that the restriction map $H^*(E) \rightarrow H^*(F)$ is surjective. Prove that $H^*(E) = H^*(F) \otimes H^*(B)$ as a $H^*(B)$ -module.

Hint. Use the Leray-Serre spectral sequence.

Exercise 3.7. Consider the fibration $\mathfrak{S} \rightarrow \mathcal{G}\tau(n)$ defined above. Prove that the restriction map $H^*(\mathfrak{S}) \rightarrow H^*(\mathfrak{F})$ is surjective.

Hint. First, prove it for \mathbb{P}^n -bundles on $\mathcal{G}\tau(n, \infty)$.

Exercise 3.8. Let B be a vector bundle over X of rank n , and $\phi : X \rightarrow \mathcal{G}\tau(n)$ the corresponding map. Consider the fibered product $X \times_{\mathcal{G}\tau(n)} \mathfrak{S}$. Prove that $X \times_{\mathcal{G}\tau(n)} \mathfrak{S}$ is isomorphic to $H^*(X) \otimes H^*(\mathfrak{F})$ as an $H^*(X)$ -module.

Hint. Use the previous exercise.

Exercise 3.9. Let B be a vector bundle over X . Prove that there exists a space Y fibered over X such that the pullback map $H^*(X) \rightarrow H^*(Y)$ is injective, and B is split on $H^*(Y)$.

Hint. Use the previous exercise.

Exercise 3.10. Prove that the natural map $\mathfrak{S} \rightarrow \mathcal{G}\tau(n)$ can be factorized through $\Phi : BU(1)^n \rightarrow \mathcal{G}\tau(n)$. Deduce that the pullback

$$\Phi^* : H^*(\mathcal{G}\tau(n)) \rightarrow H^*(BU(1)^n)$$

is injective.

Hint. To prove that it can be factorized, show that the pullback of B_{fun} to \mathfrak{S} is split, and use Exercise 3.4. Injectivity of the map $\Phi^* : H^*(\mathcal{G}\tau(n)) \rightarrow H^*(BU(1)^n)$, should follow from the injectivity of the pullback map $H^*(\mathcal{G}\tau(n), \mathbb{Q}) \rightarrow H^*(\mathfrak{S}, \mathbb{Q})$ (prove it).

Exercise 3.11. Let Σ_n denote the symmetric group, and $\frac{BU(1)^\infty}{\Sigma_\infty} := \lim_n \frac{BU(1)^n}{\Sigma_n}$

- Prove that the natural map $BU(1)^n \times BU(1)^m \rightarrow BU(1)^{n+m}$ induces the structure of H-space on $\frac{BU(1)^\infty}{\Sigma_\infty}$.
- Let B be a split bundle of rank n on X . Construct the fundamental bundle B_{fun} on $\frac{BU(1)^n}{\Sigma_n}$ and prove that there exists a map $X \rightarrow \frac{BU(1)^n}{\Sigma_n}$ such that B is a pullback of B_{fun} .
- Construct the natural map $\frac{BU(1)^\infty}{\Sigma_\infty} \rightarrow BU$, and show that it is compatible with the H-structure and injective on cohomology.
- Prove that the primitive generators of $H^{2k}(\frac{BU(1)^\infty}{\Sigma_\infty})$ are Newton polynomials $\sum z_i^k$, where z_i is the generator of $H^2(B(1))$ for a component number i in $BU(1)^n$.
- Deduce that the pullback of the primitive generator $C_k \in H^{2k}(BU)$ is proportional to $\sum z_i^k \in H^*(BU(1)^n)$.