K3 surfaces, home assignment 4: quadratic lattices

Rules: This is a class assignment for this week. Please bring your solutions (written) next Monday. We will have a class discussion the Wednesday after.

Definition 4.1. For the purposes of this assignment **a lattice** is a finitely generated torsion-free $\mathbb{Z}$-module. **Quadratic form** on a lattice is a function $q: L \longrightarrow \mathbb{Z}$, $q(l) = B(l,l)$, where B is a bilinear symmetric pairing $B: L \otimes_{\mathbb{Z}} L \longrightarrow \mathbb{Z}$. **Quadratic lattice** is a lattice equipped with a quadratic form. A quadratic form is **indefinite** if it takes positive and negative values, and **unimodular** if B is non-degenerate and defines an isomorphism $L \xrightarrow{\sim} L^*$.

Exercise 4.1. Let (L,q) be a quadratic lattice, $L_{\mathbb{Q}} := L \otimes_{\mathbb{Z}} \mathbb{Q}$, and L^* the set of all $x \in L_{\mathbb{Q}}$ such that $q(x,L) \subset \mathbb{Z}$.

a. Prove that L^* is a lattice of the same rank as L and $L \subset L^*$.

b. **The discriminant group** of L is $\mathsf{Disc}_L := L^*/L$. Prove that L is unimodular if and only if $\mathsf{Disc}_L = \{0\}$.

c. Let G be a finite abelian group. Construct a lattice (L,q) such that $\mathsf{Disc}_L = G$.

Exercise 4.2. Let (L,q) be an quadratic lattice, and $L_1 \subset L$ a sublattice.

a. Prove that $L \subset L_1^*$. Prove that any isometry $a \in O(L_1)$ takes L to another lattice $L_1 \subset a(L) \subset L_1^*$.

b. Denote by $\delta(L)$ the image of L in Disc_{L_1}. Prove that any isometry $a \in O(L_1)$ which satisfies $\delta(L) = \delta(a(L))$ preserves L.

Definition 4.2. Two subgroups $G_1, G_2 \subset GL(n,\mathbb{R})$ are called **commensurable** if $G_1 \cap G_2$ has finite index in G_1 and in G_2.

Exercise 4.3. Let (L,q) be an quadratic lattice, and $L_1 \subset L$ a sublattice. Prove that $O(L_1,q) \cap O(L,q)$ has finite index in $O(L_1,q)$.

Hint. Use the previous exercise.

Exercise 4.4. Let $nL := \bigcup_{x \in L} nx$. Prove that $nL_1 \subset L$ for any integer lattices L, L_1, and n sufficiently big. Prove that $O(nL,q) = O(L,q)$.

Exercise 4.5. Let q be a quadratic form on $L_{\mathbb{Q}} := \mathbb{Q}^n$, and $L_1, L_2 \subset L_{\mathbb{Q}}$ two lattices such that q is takes integer values on L_1, L_2. Prove that $O(L_1)$ is commensurable to $O(L_2)$.

Hint. Use Exercise 4.4 and Exercise 4.3.

Definition 4.3. Let B be the bilinear form associated with a quadratic form q. A lattice (L,B,q) is called **diagonal** if it admits an orthogonal basis, that is, a basis $z_1, ..., z_n$ such that $B(z_i, z_j) = 0$ for $i \neq j$.

Exercise 4.6. Prove that any quadratic lattice contains a diagonal sublattice of finite index.

Exercise 4.7 (*). Let (L,q) be an indefinite, non-degenerate integer lattice. Assume that either q does not represent 0 or $\operatorname{rk} L > 2$. Prove that $O(L,q)$ is infinite.

Exercise 4.8. Construct a non-degenerate integer lattice of rank 2, not commensurable to a unimodular lattice.