K3 surfaces, assignment 8: quaternionic Hermitian structures

Definition 8.1. An **almost hypercomplex structure** on a manifold M is a triple almost complex structures (I, J, K) satisfying the quaternionic relations. It is called **hypercomplex** if I, J, K are integrable. **An almost hypercomplex quaternionic Hermitian structure** on M is an almost hyper complex structure (I, J, K) and a Riemannian metric h which is invariant under the action of I, J, K.

Exercise 8.1. Let (M, I, J, K, g) be an almost hypercomplex quaternionic Hermitian manifold, and $\omega_J := g(J\cdot, \cdot), \omega_K := g(K\cdot, \cdot)$ its fundamental forms. Prove that $\omega_J + \sqrt{-1}\omega_K \in \Lambda^{2,0}(M, I)$.

Exercise 8.2. Let (M, I) be almost complex manifold.

a. (*) Prove that for any non-degenerate (2,0)-form Ω on (M, I) there exists an almost hypercomplex Hermitian structure (M, I, J, K, g) such that $\Omega = \omega_J + \sqrt{-1}\omega_K$.

b. Prove that such a hypercomplex Hermitian structure (M, I, J, K, g) is not unique.

Definition 8.2. Let M be an oriented Riemannian 4-manifold, and $*: \Lambda^i(M) \longrightarrow \Lambda^{4-i}(M)$ the Hodge $*$ operator. The bundle $\{\eta \in \Lambda^2(M) \mid *\eta = \eta\}$ is called **the bundle of the autodual 2-forms** and denoted $\Lambda^+(M)$. The bundle $\{\eta \in \Lambda^2(M) \mid *\eta = -\eta\}$ is called **the bundle of the anti-autodual 2-forms** and denoted $\Lambda^-(M)$.

Exercise 8.3. Let (M, g) be an oriented Riemannian 4-manifold.

a. Prove that $\Lambda^2(M) = \Lambda^+(M) \oplus \Lambda^-(M)$, and both $\Lambda^+(M)$ and $\Lambda^-(M)$ are 3-dimensional.

b. Prove that the decomposition $\Lambda^2(M) = \Lambda^+(M) \oplus \Lambda^-(M)$ depends only on the conformal class of the metric g.

c. Let M be an oriented 4-dimensional manifold, Vol its volume form and $\Lambda^2(M) = A \oplus B$ be a decomposition onto a direct sum of 2 bundles of rank 3. Assume that the scalar product $h(x, y) := \frac{x \wedge y}{\text{Vol}}$ is positive definite on A, negative definite on B, and A is orthogonal to B. Prove that there exists a Riemannian metric on M such that $A = \Lambda^+(M)$ and $B = \Lambda^-(M)$. Prove that such a metric is unique up to a conformal multiplier.

Exercise 8.4. Let (M, I, J, K, g) be an almost hypercomplex quaternionic Hermitian manifold, $\dim_{\mathbb{R}} M = 4$. Consider the natural action of $SU(2) = U(\mathbb{H}, 1) = Sp(1)$ on the tangent bundle TM. Denote by $\Lambda^2(M)^{SU(2)}$ the bundle of $SU(2)$-invariant antisymmetric 2-forms.

a. Prove that $\Lambda^+(M) = \langle \omega_I, \omega_J, \omega_K \rangle$.

b. Prove that $\Lambda^-(M) = \Lambda^2(M)^{SU(2)}$.

c. Prove that $\Lambda^2(M)^{SU(2)} = \Lambda^{1,1}(M,I) \cap \Lambda^{1,1}(M,J) \cap \Lambda^{1,1}(M,K)$

d. Prove that $\Lambda^2(M)^{SU(2)}$ is the bundle of all $\eta \in \Lambda^{1,1}(M,I)$ which satisfy $\eta \wedge \omega_I = 0$.

e. Prove that (c) is true for any dimension of M, and (d) is false when $\dim_{\mathbb{R}} M > 4$.

Exercise 8.5. Let (M,I,J,K) be an almost hypercomplex manifold, $\dim_{\mathbb{R}} M = 4$. Prove that M admits a quaternionic Hermitian form, which is defined uniquely up to a conformal multiplier.

Hint. Use the previous exercise and Exercise 8.3.

Exercise 8.6 (*). Let $\omega_1, \omega_2, \omega_3$ be a triple of 2-forms on a 4-manifold M. Assume that the intersection matrix $a_{ij} = \frac{\omega_i \wedge \omega_j}{\omega_1^2}$ is positive definite and has constant coefficients. Prove that M admits an almost hypercomplex quaternionic Hermitian structure, and all ω_i are linear combinations of the fundamental forms $\omega_I, \omega_J, \omega_K$.

Exercise 8.7. Let (M,I,J,K) be an almost hypercomplex manifold. Consider the natural action of $SU(2) = U(\mathbb{H},1) = Sp(1)$ on the tangent bundle TM. Denote by $\Lambda^2(M)^{SU(2)}$ the bundle of $SU(2)$-invariant antisymmetric 2-forms. Prove that $\Lambda^2(M)^{SU(2)} = \Lambda^{1,1}(M,I) \cap \Lambda^{1,1}(M,J) \cap \Lambda^{1,1}(M,K)$.

Exercise 8.8 (*). Let (M,I,J,K) be an almost hypercomplex manifold. Consider the natural action of $SU(2) = U(\mathbb{H},1) = Sp(1)$ on the tangent bundle TM. Denote by $\Lambda^2(M)^{SU(2)}$ and $\mathsf{Sym}^2(M)^{SU(2)}$ the bundles of $SU(2)$-invariant antisymmetric and symmetric 2-forms. Let $\Lambda^{1,1}(M,I)^+$ denote all (1,1)-forms on (M,I) which belong to non-trivial irreducible representations of $SU(2)$. Prove that the following bundles are naturally isomorphic:

a. $\Lambda^2(M)^{SU(2)} = \mathsf{Sym}^{2,0}(M,I)$

b. $\Lambda^{1,1}(M,I)^+ = \Lambda^{2,0}(M,I)$

Exercise 8.9 (*). Let (M,I) be almost complex manifold, A the (infinitely-dimensional) space of all non-degenerate (2,0) forms, and B the space of all almost hypercomplex Hermitian structures (I,J,K,h). Prove that A and B are homotopy equivalent.

Exercise 8.10 (!). Let V be a complex 2n-dimensional Hermitian vector space with orthonormal basis $p_1,...,p_n,q_1,...,q_n$, and $\Omega = \sum_{i=1}^n p_i^* \wedge q_i^*$ a complex linear symplectic form. Denote by $Sp(2n,\mathbb{C})$ the group of matrices preserving Ω. Prove that there exists a quaternionic algebra action on V such that $Sp(2n,\mathbb{C}) \cap U(2n) = U(n,\mathbb{H})$, where $U(n,\mathbb{H})$ is the group of matrices preserving the metric and the quaternionic action.