K3 surfaces, 2024, exam

Rules: Rules: Every student receives from me a list of 12 exercises (chosen randomly), and has to solve as many of them as you can before March 2025. Please write down the solution and bring it to exam for me to see. The final score N is obtained by summing up the points from the exam problems and the class tests, using the formula $N = e + t/2$, where t is the sum of class tests, e the points for exam problems. Marks: C when $30 \leqslant N < 45$, B when $45 \leqslant N < 70$, A when $70 \leqslant N \leqslant 90$, A+ when $N > 90$.

1 Differential forms on a K3 surface

Exercise 1.1 (10 points). Let $Z \subset (M, \Omega)$ be a 2-dimensional real submanifold in a K3 surface. Assume that Z is Lagrangian with respect to the symplectic forms $Re\,\Omega$ and $Im\,\Omega$. Prove that Z is complex analytic.

Exercise 1.2 (30 points). Let ω_1, ω_2 be symplectic forms on a K3 surface, satisfying $\omega_1 \wedge \omega_2 = 0$. Prove that $\omega_1^2 = \omega_2^2$, or find a counterexample.

Definition 1.1. A holomorphic Lagrangian fibration on a holomorphically symplectic manifold (M, Ω) is a holomorphic submersion $\pi : M \longrightarrow X$ such that the fibers of π are holomorphic Lagrangian with respect to Ω.

Exercise 1.3 (10 points). Let $\pi : M \longrightarrow X$ be a holomorphic Lagrangian fibration on a K3 surface, and $\sigma : X \longrightarrow M$ a smooth section. Prove that $\sigma^*(\Omega)$ has Hodge type $(2,0) + (1,1)$.

Exercise 1.4 (20 points). Let $\omega_1, \omega_2, \omega_3$ be a triple of 2-forms on a 4-manifold M. Assume that the intersection matrix $a_{ij} = \frac{\omega_i \wedge \omega_j}{\omega_1^2}$ is positive definite and has constant coefficients. Prove that M admits a hyperkähler structure, such that all ω_i are linear combinations of the fundamental forms $\omega_I, \omega_J, \omega_K$.

Exercise 1.5 (20 points). Let M be a compact hypercomplex manifold of real dimension 4, equipped with a quaternionic Hermitian structure, and V the space of closed $SU(2)$-invariant 2-forms. Prove that V is finite-dimensional.

Exercise 1.6. Let (M, I, J, K, g) be an almost hypercomplex Hermitian structure on a K3 surface, and $\omega_I, \omega_J, \omega_K$ its fundamental forms.

a. (20 points) Suppose that ω_I is closed. Prove that ω_J, ω_K are closed, or find a counterexample.

b. (30 points) Suppose that ω_I, ω_J are closed. Prove that ω_K is closed, or find a counterexample.

2 Complex curves on a K3 surface

Exercise 2.1 (10 points). Construct a Kummer surface which contains two intersecting rational curves.

Exercise 2.2 (20 points). Let L be an ample bundle on a K3 surface M. Prove that $L^{\otimes 2}$ is globally generated (that is, for each $x \in M$ there exists a section $h \in H^0(L^{\otimes 2})$ which does not vanish in x).

Exercise 2.3 (10 points). Let $\pi : M \longrightarrow X$ be a holomorphic Lagrangian fibration on a K3 surface with connected fibers. Prove that the general fiber of π is an elliptic curve, and the base is a rational curve.

Exercise 2.4 (10 points). Let $G = \mathbb{Z}/p$ be a cyclic group of prime order acting on a K3 surface by holomorphically symplectic automorphisms. Prove that the action of G is either trivial or free outside of a finite set.

Exercise 2.5. Let $\phi : M \longrightarrow \mathbb{C}P^2$ be a finite holomorphic map, and $S \subset \mathbb{C}P^2$ its ramification divisor. Assume that S is smooth.

a. (10 points) Prove that S is a sextic.

b. (20 points) Let $S \subset \mathbb{C}P^2$ be a smooth sextic. Construct a K3 surface M and a finite map $\phi : M \longrightarrow \mathbb{C}P^2$ ramified in S.

Exercise 2.6. Let S be a smooth genus g curve which can be embedded in a K3 surface M, and X the family of all deformations of S in M.

a. (10 points) Prove that $\dim X \leqslant g$.

b. (30 points) Let $\mathcal{X}_g$ be the space of all curves of genus g which can be possibly embedded to a K3 surface. Prove that each irreducible component Z of $\mathcal{X}_g$ satisfies $\dim_{\mathbb{C}} Z \leqslant g + 19$. Deduce that there exists a compact complex curve which cannot be embedded to a K3 surface.

Exercise 2.7 (10 pt). Let $S \subset M$ be a singular, irreducible complex curve on a K3 surface, and $\tilde{S}$ its normalization. Prove that $g(\tilde{S}) \geqslant 0$.

3 Kähler cone, periods and automorphisms

Exercise 3.1 (20 points). Let M be a K3 surface such that $\operatorname{rk} NS(M) < 20$. Using Demailly-Păun theorem, prove that there exists a non-zero vector $\eta H^{1,1}(M)$ which satisfies $\eta^2 = 0$ and belongs to the boundary of the Kähler cone of M.

Exercise 3.2 (20 points). Prove that there exists a K3 surface of algebraic dimension 1.

Exercise 3.3 (20 points). Let ϕ be an automorphism of a K3 surface preserving a vector $v \in H^{1,1}(M, \mathbb{R})$ such that $\int_M v \wedge v > 0$. Prove that ϕ has finite order.

Exercise 3.4 (30 points). Let ϕ be an automorphism of an algebraic K3 surface. Prove that the action of ϕ on $H^{2,0}(M)$ has finite order.

Exercise 3.5 (30 points). Let ϕ be an automorphism of an algebraic K3 surface, and A_n is the operator norm of ϕ^n acting on $H^2(M)$. Assume that there exists a constant C such that $A_n \leqslant C < \infty$. Prove that ϕ has finite order.

Exercise 3.6 (20 points). Let M be a K3 surface with Picard rank 2 and an infinite automorphism group. Prove that M does not contain a (-2)-curve.

Exercise 3.7 (30 points). Let M be a K3 surface with Picard rank 3 containing (-2)-curve. Prove that $\operatorname{Aut}(M)$ has a finite index cyclic subgroup, or find a counterexample.