K3 surfaces

lecture 2: Classifying spaces

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Bialgegras

DEFINITION: Let $A^\bullet$, $B^\bullet$ be graded commutative algebras The **tensor product algebra** is $A^\bullet \otimes B^\bullet$ with the product $a \otimes b \cdot a' \otimes b' = (-1)^{\tilde{b}\tilde{a}'} aa' \otimes bb'$.

REMARK: By Künneth formula, $H^\bullet(X \times Y)$ is isomorphic to $H^\bullet(X) \otimes H^\bullet(Y)$ as an algebra.

DEFINITION: Let $A^\bullet$ be a graded commutative algebra over a field $\mathbb{K}$. We say that $A^\bullet$ is a **bialgebra** if it is equipped with a homomorphism of algebras $A \xrightarrow{\Delta} A \otimes A$, called **comultiplication** which is **coassociative**, that is, satisfies

$$\Delta \circ \Delta \otimes \operatorname{Id}_A = \Delta \circ \operatorname{Id}_A \otimes \Delta : \ A \longrightarrow A \otimes_{\mathbb{K}} A \otimes_{\mathbb{K}} A.$$

Counit of a bialgebra is an algebra homomorphism $A \xrightarrow{\varepsilon} \mathbb{K}$ which satisfies $\Delta \circ (\varepsilon \otimes \operatorname{Id}_A) = \Delta \circ (\operatorname{Id}_A \otimes \varepsilon) = \operatorname{Id}_A$ In the sequel, we shall tacitly assume that all bialgebras have counit.

REMARK: **Coassociative comultiplication means that the dual space $(A^\bullet)^*$ is equipped with an algebra structure.** Compatibility of comultiplication with the multiplication in $A^\bullet$ means that **this algebra structure on $(A^\bullet)^*$ is a morphism of A-modules.**

Examples of bialgebras

EXAMPLE: Let N be a set equipped with an associative operation $N \times N \xrightarrow{m} N$ with unit e (such a structure is called **the structure of a monoid**, or **semigroup with unit** Then **the ring of $\mathbb{K}$-valued functions $C(N)$ is a bialgebra**, with comultiplication morphism given by $m^* : C(N) \longrightarrow C(N \times N) = C(N) \otimes_{\mathbb{K}} C(N)$, and counit $\varepsilon(v) = v(e)$.

REMARK: The notion of a bialgebra is an abstraction of this observation: heuristically speaking, **bialgebras are alrebras of functions on semigroups**.

EXAMPLE: Let N be a topological space equipped with a continuous map $N \times N \xrightarrow{m} N$ inducing the structure of a monoid. Consider the comultiplication on the cohomology algebra $H^\bullet(N)$, given by $m^* : H^\bullet(N) \longrightarrow H^\bullet(N \times N) = H^\bullet(N) \otimes_{\mathbb{K}} H^\bullet(N)$. **Then $H^\bullet(N)$ is a bialgebra**. Indeed, coassociativity of m^* follows from associativity of N, and counit is given by the pullback to $H^\bullet(e) = H^0(e) = \mathbb{K}$.

H-spaces

DEFINITION: An H-space is a topological space M equipped with a continuous map $M \times M \xrightarrow{\mu} M$ (“the multiplication map”) and an element $e \in M$ (“the unit”) which satisfy “semigroup conditions up to homotopy”, namely the following.

* **Homotopy associativity:** the maps $\mu \times \operatorname{Id} \circ mu: \ M \times M \times M \longrightarrow M$ and $\operatorname{Id} \times \mu \circ \mu: \ M \times M \times M \longrightarrow M$ are homotopic.

* **Homotopy unit:** the map $\mu: \ M \times \{e\} \longrightarrow M$ is homotopic to identity.

EXAMPLE: Clearly, **any toplogical group is an H-space.**

CLAIM: Let M be an H-space. **Then the cohomology algebra $H^\bullet(M)$ is a bialgebra.**

Proof: The comultiplication map $H^*(M) \longrightarrow H^*(M) \otimes H^*(M) = H^*(M \times M)$ is induced by the multiplication $M \times M \xrightarrow{\mu} M$. ■

Loop spaces as H-spaces

EXAMPLE: Let $\Omega(M, x)$ be the space of loops, that is, paths $\gamma : [0,1] \longrightarrow M$ starting and ending in x. We can multiply loops by mapping a pair $\gamma_1, \gamma_2 : [0,1] \longrightarrow M$ to a loop $\gamma_1\gamma_2$ $[0,1] \longrightarrow M$ equal to $\gamma_1(2t)$ on $[0,1/2]$ and to $\gamma_2(2t-1)$ on $[1/2,1]$. The homotopy unit is the constant loop. This defines **the structure of an H-group on the loop space**.

REMARK: The topology on the loop space $\Omega(M, x)$ can be defined, for example, **by assuming that M is a metric space, and consider the uniform topology on the maps $\gamma : [0,1] \longrightarrow M$.** In more generality, we take **the compact-open topology**, with the base sets consisting of all loops which map a given compact $K \subset [0,1]$ to an open set $U \subset M$.

Bialgebras of finite type

Let $V^\bullet$ be a graded vector space. Denote by $\operatorname{Sym}_{gr}(V^\bullet)$ the tensor product $\operatorname{Sym}^*(V^{\text{even}}) \otimes \Lambda^*(V^{\text{odd}})$ with a natural grading. On $\operatorname{Sym}^*(V^{\text{even}}) \otimes \Lambda^*(V^{\text{odd}})$ one has a natural structure of an algebra.

DEFINITION: Free commutative algebra is a polynomial algebra. **Free graded commutative algebra** is $Sym_{gr}(V^\bullet)$, where $V^\bullet$ is a graded vector space.

DEFINITION: A graded algebra of finite type is an algebra graded by $i \geqslant 0$, with all graded components finitely-dimensional.

THEOREM: (Hopf theorem) Let $A^\bullet$ be a graded bialgebra of finite type over a field $\mathbb{K}$ of characteristic 0. **Then $A^\bullet$ is a free graded commutative $\mathbb{K}$-algebra**.

Proof: Next lecture. ■

COROLLARY: For any Lie group $H^*(G)$ is finite-dimensional, hence **$H^*(G)$ is isomorphic to a Grassmann algebra.** In particular, $\dim H^*(G) = 2^n$. ■

Heinz Hopf (1894-1971)

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Cohomology algebra of $U(n)$

CLAIM: The cohomology algebra of $H^*(U(n),\mathbb{Q})$ – **is a free graded commutative algebra with generators in degrees $1,3,5,...,2n-1$.**

Proof. Step 1: Since $U(n)$ is a Lie group, its cohomology is a Hopf algebra. Therefore, $H^*(U(n-1))$ **is a free graded commutative algebra with generators in odd degrees.** Using induction, we can assume that $H^*(U(n-1))$ – **is a free algebra with generators in degrees $1,3,5,...,2n-3$.**

Step 2: The group $U(n)$ is fibered over S^{2n-1} with fiber $U(n-1)$. The corresponding Leray-Serre spectral sequence has $E_2^{p,q} = H^p(S^{2n-1})\otimes H^q(U(n-1))$. Generators of $H^q(U(n-1))$ are of degree $1,3,5,...,2n-3$, hence their differential, which belongs to $H^{>0}(S^{2n-1})\otimes H^q(U(n-1))$, vanishes. Then $H^{2n-1}(S^{2n-1})$ is another generator of the free algebra $H^*(U(n))$, in addition to the generators of $H^*(U(n-1))$, which implies that the spectral sequence $H^p(S^{2n-1})\otimes H^q(U(n-1)) \Rightarrow H^{p+q}(U(n))$ degenerates in E^2. ■

Grassmann manifolds

DEFINITION: Let $\operatorname{Gr}(n,m)$ be the **Grassmann manifold**, or **Grassmannian**, that is, the space of n-dimensional planes in $V=\mathbb{K}^m$, where $\mathbb{K}=\mathbb{R}$ or $\mathbb{C}$. The **fundamental bundle** B_{fun} is a rank n vector bundle over $\operatorname{Gr}(n,m)$, with the fiber W at each point $W\in\operatorname{Gr}(n,m)$.

REMARK: Let B_{triv} be the trivial rank m vector bundle over the Grassmannian $\operatorname{Gr}(n,m)$. Then B_{fun} is naturally embedded to B_{triv}. After equipping B_{triv} with a (Hermitian if $\mathbb{K}=\mathbb{C}$ or Euclidean when $\mathbb{K}=\mathbb{R}$) metric and taking an orthogonal complement, **we obtain a decomposition $B_{triv}=B_{\text{fun}}\oplus B_{\text{fun}}^\perp$.**

REMARK: Let B be a rank n bundle on X, B' a rank $n-m$ bundle, such that $B\oplus B'$ is trivial. Identifying all fibers of $B\oplus B'$ with a vector space $V=\mathbb{K}^m$, we obtain that every $x\in X$ defines a subspace $B|_x\subset(B\oplus B')|_x=V$.

This proves the following claim.

Claim 1: Let B, B' be a rank n and $m-n$ vector bundles over a space X such that $B\oplus B'$ is trivial. **Then there exists a map $\varphi: X\longrightarrow\operatorname{Gr}(n,m)$ such that $\varphi^*B_{\text{fun}}\cong B$.** ■

Grassmannian $\operatorname{Gr}(n, \infty)$

DEFINITION: Considrer the natural embeddings

$$\operatorname{Gr}(n, m) \hookrightarrow \operatorname{Gr}(n, m+1) \hookrightarrow \operatorname{Gr}(n, m+2) \hookrightarrow ...,$$

associated with the maps $\mathbb{K}^m \hookrightarrow \mathbb{K}^{m+1} \hookrightarrow \mathbb{K}^{m+2} \hookrightarrow$ ... and **let $\operatorname{Gr}(n) := \operatorname{Gr}(n, \infty)$ denote the union of all these spaces** (that is, the inductive limit of cellular complexes). By definition, $\operatorname{Gr}(n)$ is the space of n-dimensional subspaces in $\mathbb{K}^\infty$, where $\mathbb{K}^\infty$ denotes. $\bigoplus_{i=0}^\infty \mathbb{K}$.

THEOREM: Let B be a vector bundle over a manifold M (paracompact with countable base). **Then $B \oplus B'$ is trivial, for some vector bundle B' over M.**

EXERCISE: Prove this theorem **(it is proven the same way as Whitney embedding theorem, and for the same class of manifolds).**

COROLLARY: Let B be a rank n vector bundle on a manifold M. **Then there exists a map $\varphi : M \longrightarrow \operatorname{Gr}(n)$ such that $B = \varphi^* B_{\mathsf{fun}}$.**

Proof: Using the previous theorem, we obtain a trivialization of a bundle $B \oplus B'$ for some vector bundle B' on M. **The corresponding map $\varphi : M \longrightarrow \operatorname{Gr}(n, m)$ is constructed in Claim 1.** ■

Classifying spaces

DEFINITION: Let G be a topological group, and X a topological space. **A principal G-bundle** is a topological space E equipped with a free action of G such that $E/G = X$. In this case, X is called **the total space** of the principal G-bundle $E \longrightarrow X$.

DEFINITION: A classifying space for a topological group G is a topological space BG, equipped with a principal G-bundle, such that its total space is contractible.

DEFINITION: A G-space is a topological space equipped with a G-action. **Morphism** of G-spaces is a continuous map compatible with G-action. Two G-morphisms $\varphi_1, \varphi_0 : A \longrightarrow B$ are called **G-homotopic** if there exists a G-morphism $\Phi : [0,1] \times A \longrightarrow B$ such $\Phi\big|_{\{0\}\times A} = \varphi_0$ and $\Phi\big|_{\{1\}\times A} = \varphi_1$. In other words, a G-homotopy is a homotopy which is a G-morphism. A **G-homotopy equivalence** is a homotopy equivalence which is a G-morphism.

THEOREM: (Atiyah, Bott)
A classifying space BG is unique up to homotopy equivalence.

Proof: This would follow if we prove that **a contractible space with free G-action is unique up to a G-homotopy equivalence;** this is left as an exercise. ■

Classifying space and homotopy classes of G-bundles

DEFINITION: Let E be a contractible space with a free G-action, and $BG = E/G$. The **fundamental G-bundle** over BG is E, considered as a G-bundle.

THEOREM: Let X be a cellular space. Then **the homotopy classes of maps $X \longrightarrow BG$ are in bijective correspondence with the isomorphism classes of principal G-bundles over X:** for each G-bundle Y there exists a map $X \xrightarrow{\varphi} BG$ such that $\varphi^* G_{\text{fun}} \cong Y$.

Proof: Left as an exercise. **We will prove this result when $G = U(n)$ or $G = O(n)$ and $BG = \operatorname{Gr}(n, \infty)$.** ■

REMARK: For Grassmannian **this is actually already proven.** As we have shown, the vector bundles, or, equivalently, principal frame bundles are all induced by the maps $X \longrightarrow \operatorname{Gr}(n, \infty)$. However, **we did not prove yet that $BU(n) = \operatorname{Gr}_{\mathbb{C}}(n, \infty)$ and $BO(n) = \operatorname{Gr}_{\mathbb{R}}(n, \infty)$.**

Stiefel spaces

DEFINITION: Fix a Hermitian or Euclidean metric on $\mathbb{K}^\infty$. Let $\mathrm{Gr}(n,\infty)$ be the Grassmannian, B_{fun} its fundamental bundle, and $\mathrm{St}(n,\infty)$ the space of orthonormal frames in B_{fun}. It is called **the $GL(n,\mathbb{K})$-Stiefel space**.

REMARK: Clearly, there is a free $GL(n,\mathbb{K})$-action on $\mathrm{St}(n,\infty)$, and the quotient is $\mathrm{Gr}(n,\infty)$. In other words, **the Stiefel space is the total space of a principal $U(n,\mathbb{C})$-bundle or $O(n,\mathbb{R})$-bundle over $\mathrm{Gr}_{\mathbb{K}}(n,\infty)$.**

EXERCISE: Prove that $\mathrm{St}(n,\infty)$ is contractible.

This implies that $\mathrm{Gr}(n,\infty)$ is the classifying space for $G = U(n)$ or $G = O(n)$.

Stiefel spaces are contractible

THEOREM: The Stiefel space $\operatorname{St}(n,\infty)$ is contractible, and therefore the Grassmanian $\operatorname{Gr}(n,\infty)$ is the classifying space for $G = U(n)$ or $G = O(n)$.

Proof. Step 1: Let $Y \longrightarrow X$ be a locally trivial fibration such that its fiber is contractible and the base is also contractible. **Then Y is contractible. Prove this!**

Step 2: Let $S^\infty \subset \mathbb{K}^\infty$ be a unit sphere in $\mathbb{K}^\infty$, where $\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$. **Then $\operatorname{St}(n,\infty)$ is fibered over $\operatorname{St}(n-1,\infty)$ with the fiber S^∞.** To prove that $\operatorname{St}(n,\infty)$ is contractible, it remains to show that S^∞ is contractible and use induction in n. **The induction base is clear because $\operatorname{St}(1,\infty) = S^\infty$.**

Step 3: Choose a basis $\{z_i\}$ in $\mathbb{R}^\infty$ numbered by $\mathbb{Z}^{\geqslant 0}$, and let $R: \mathbb{R}^\infty \longrightarrow \mathbb{R}^\infty$ take z_i to z_{i+1}, and $R_t := t\operatorname{Id} + (1-t)R$. Then $z \mapsto \frac{R_t(z)}{|R_t(z)|}$ takes S^∞ to itself for all $t \in \mathbb{R}$, **hence R is homotopic to identity.**

Step 4: This gives a homotopy between identity and a map which takes a sphere to a sphere without a point. However, the sphere without a point is contractible, **hence the identity is homotopic to a map contracting the sphere to a point.** ■

The infinite Grassmannian

EXERCISE: Let $X_\infty = \bigcup X_i$ be an inductive limit of contractible cellular spaces. **Prove that X_∞ is also contractible.**

DEFINITION: Choose a basis $x_0, x_1, ...,$ in $\mathbb{C}^\infty$ or $\mathbb{R}^\infty$, and let $R: \mathbb{C}^\infty \longrightarrow \mathbb{C}^\infty$ be defined as above, $R(x_i) = x_{i+1}$. Consider the embedding $Gr(n,\infty) \hookrightarrow Gr(n+1,\infty)$, taking the space $L \subset \mathbb{C}^\infty$ to $\langle x_0, R(L)\rangle$. The union (inductive limit) $\bigcup_n Gr(n,\infty)$ is called **the infinite Grassmannian**, and is denoted as BU.

THEOREM: **BU is the classifying space for $U = \bigcup_n U(n)$.**

Proof: Consider the embeddings $\operatorname{St}(1,\infty) \hookrightarrow \operatorname{St}(2,\infty) \hookrightarrow ...$ where an orthonormal frame $\{\zeta_1, ..., \zeta_m\} \in \mathbb{C}^\infty$ is mapped to $\{x_0, R(\zeta_1), ..., R(\zeta_m)\}$, and let $\operatorname{St}(\infty,\infty)$ be their union. By construction, $BU = \operatorname{St}(\infty,\infty)/U(\infty)$, hence **it is sufficient to prove that $\operatorname{St}(\infty,\infty)$ is contractible.** This follows from the previous exercise. ■

Stable equivalence

DEFINITION: Vector bundles B_1, B_2 are called **stably equivalent** if $B_1 \oplus U_1 \cong B_2 \oplus U_2$, where U_i are trivial vector bundles.

THEOREM: Let X be a finite cellular space. Then **homotopy classes of maps $X \longrightarrow BU$ are in bijective correspondence with classes of stable equivalence of vector bundles.**

Proof: Left as an exercise. ■

BU as an H-space

Bott periodicity identifies the space of loops on U and BU; this implies that BU is an H-space (loop spaces are H-spaces). However, the H-structure on BU can be constructed explicitly.

PROPOSITION: Consider a map $S: \mathbb{C}^\infty \times \mathbb{C}^\infty \longrightarrow \mathbb{C}^\infty$ taking the basis vectors x_i of the first space to x_{2i} and the basis vectors of the second space to x_{2i+1}. Then $L, L' \longrightarrow S(L, L')$ **defines a structure of an H-space on the infinite Grassmannian BU.**

Proof: We need to show H-associativity, which would follow if we prove that any permutation of basis elements is homotopic to identity. This is left as an exercise. ∎

COROLLARY: **$H^*(BU, \mathbb{Q})$ is a free supercommutative algebra.**

Proof: Follows from Hopf theorem. ∎

REMARK: It is not hard to write a celular decomposition for the Grassmannian ("Schubert cells"); all cells are even-dimensional, which gives the dimensions of the groups $H^i(BU)$. Then the Hopf theorem can be used to compute the cohomology algebra. This also implies that **the algebra $H^*(BU)$ is commutative (and therefore, free polynomial).**