K3 surfaces

lecture 3: Hopf theorem

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Bialgegras (reminder)

DEFINITION: Let $A^\bullet$, $B^\bullet$ be graded commutative algebras The **tensor product algebra** is $A^\bullet \otimes B^\bullet$ with the product $a \otimes b \cdot a' \otimes b' = (-1)^{\tilde{b}\tilde{a}'} aa' \otimes bb'$.

REMARK: By Künneth formula, $H^\bullet(X \times Y)$ is isomorphic to $H^\bullet(X) \otimes H^\bullet(Y)$ as an algebra.

DEFINITION: Let $A^\bullet$ be a graded commutative algebra over a field k. We say that $A^\bullet$ is a **bialgebra** if it is equipped with a homomorphism of algebras $A \stackrel{\Delta}{\longrightarrow} A \otimes A$, called **comultiplication** which is **coassociative**, that is, satisfies

$$\Delta \circ \Delta \otimes \operatorname{Id}_A = \Delta \circ \operatorname{Id}_A \otimes \Delta : \ A \longrightarrow A \otimes_k A \otimes_k A.$$

Counit of a bialgebra is an algebra homomorphism $A \stackrel{\varepsilon}{\longrightarrow} k$ which satisfies $\Delta \circ (\varepsilon \otimes \operatorname{Id}_A) = \Delta \circ (\operatorname{Id}_A \otimes \varepsilon) = \operatorname{Id}_A$ In the sequel, we shall tacitly assume that all bialgebras have counit.

REMARK: **Coassociative comultiplication means that the dual space $(A^\bullet)^*$ is equipped with an algebra structure.** Compatibility of comultiplication with the multiplication in $A^\bullet$ means that **this algebra structure on $(A^\bullet)^*$ is given by a morphism of A-modules.**

Bialgebras of finite type (reminder)

Let $V^\bullet$ be a graded vector space. Denote by $\operatorname{Sym}_{gr}(V^\bullet)$ the tensor product $\operatorname{Sym}^*(V^{\text{even}}) \otimes \Lambda^*(V^{\text{odd}})$ with a natural grading. On $\operatorname{Sym}^*(V^{\text{even}}) \otimes \Lambda^*(V^{\text{odd}})$ one has a natural structure of an algebra.

DEFINITION: Free commutative algebra is a polynomial algebra. **Free graded commutative algebra** is $Sym_{gr}(V^\bullet)$, where $V^\bullet$ is a graded vector space.

DEFINITION: A graded algebra of finite type is an algebra graded by $i \geqslant 0$, with all graded components finitely-dimensional.

THEOREM: (Hopf theorem) Let $A^\bullet$ be a graded bialgebra of finite type over a field k of characteristic 0. **Then $A^\bullet$ is a free graded commutative k-algebra**.

REMARK: This allows one to compute the multiplicative structure **on all Lie groups and on all loop spaces of finite CW-spaces.**

REMARK: For any Lie group $H^*(G)$ is finite-dimensional, hence **$H^*(G)$ is isomorphic to a Grassmann algebra.** In particular, $\dim H^*(G) = 2^n$.

Heinz Hopf (1894-1971)

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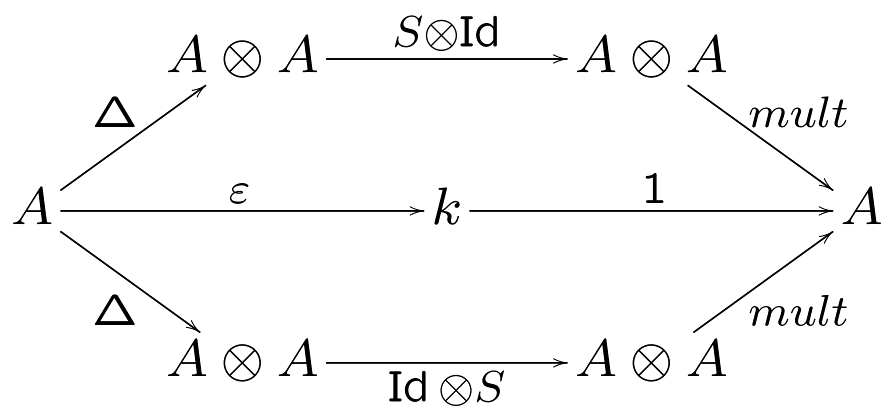
Hopf algebras

DEFINITION: A bialgebra is called **a Hopf algebra** if it is equipped with a homomorphism $A \xrightarrow{S} A$ **("the antipode map")**, and the following diagram is commutative:



REMARK: The antipode condition is self-dual: if A is a Hopf algebra, the dual space A^* is also a Hopf algebra, multiplication goes to comultiplication.

EXAMPLE: Let N be a group, and $C(N)$ the space of functions on N equipped with the bialgebra structure Then the map $n \longrightarrow n^{-1}$ defines an antipode structure on $C(N)$. We obtain that **the algebra $C(N)$ of functions on a group is a Hopf algebra** (check this).

EXAMPLE: Let G be a topological group, and $H^*(G)$ its cohomology algebra. Consider the map $H^*(G) \xrightarrow{S} H^*(G)$, induced by $x \longrightarrow x^{-1}$. **Then $H^*(G)$ is a Hopf algebra** (check this).

Primitive elements in a bialgebra

DEFINITION: An element x of a bialgebra is called **primitive** if $\Delta(x) = x \otimes 1 + 1 \otimes x$.

REMARK: First, the Hopf theorem is proven for all bialgebras, generated (multiplicatively) by the primitive elements, and then we prove that finite type bialgebras are generated by primitive elements.

DEFINITION: Let A be a bialgebra, and $P \subset A$ the space of primitive elements. Consider the natural homomorphism $\operatorname{Sym}_{gr}(P) \xrightarrow{\psi} A$. We say that A **is free up to degree k** if $\bigoplus^{i \leqslant k} \operatorname{Sym}^i_{gr}(P) \xrightarrow{\psi} A$ is an embedding.

REMARK: The following lemma **immediately implies Hopf theorem for all bialgebras generated by primitive elements.**

LEMMA: Let $A^\bullet$ be a bialgebra which is free up to degree k. **Then $A^\bullet$ is free up to degree $k+1$.**

Hopf theorem for bialgebras generated by primitive elements

LEMMA: Let $A^\bullet$ be a bialgebra which is free up to degree k. **Then $A^\bullet$ is free up to degree $k+1$.**

Proof. Step 1: Let $\{x_i\}$ be a basis in the space P of primitive elements. Consider a polynomial relation of degree $k+1$, say, $Q(x_1, ..., x_n) = 0$, and represent it as a polynomial of x_1 with coefficients which are polynomials of $x_2, ..., x_n$: $Q = Q_m x_1^m + Q_{m-1} x_1^{m-1} + ... + Q_0$. Clearly, $\Delta(Q) = Q \otimes 1 + 1 \otimes Q + R$, where $R \in \mathfrak{A} := \left(\bigoplus^{i \leqslant k} \operatorname{Sym}^i_{gr}(P)\right) \otimes \left(\bigoplus^{i \leqslant k} \operatorname{Sym}^i_{gr}(P)\right)$.

Step 2: Since $\psi : \bigoplus^{i \leqslant k} \operatorname{Sym}^i_{gr}(P) \xrightarrow{\psi} A$ is an embedding, and elements of $\mathfrak{A}$ belong to $\operatorname{im}\psi \otimes \operatorname{im}\psi$, each element of $\mathfrak{A}$ can be uniquely represented as a sum of monomials $\lambda \otimes \mu$, where λ, μ are degree $\leqslant k$ monomials on x_i. Denote by $\Pi : \mathfrak{A} \longrightarrow x_1 \otimes \left(\bigoplus^{i \leqslant k} \operatorname{Sym}^i_{gr}(P)\right)$ the projection to the sum of all monomials of form $x_1 \otimes \mu$. Since $\Delta(x_i) = x_i \otimes 1 + 1 \otimes x_i$, one has $\Delta(x_1^m) = (x_1 \otimes 1 + 1 \otimes x_1)^m$, **giving $\Pi(\Delta(x_1^m)) = m x_1 \otimes x_1^{m-1}$.**

Step 3: Let $\Pi(R) := x_1 \otimes R_0$. Since $Q = 0$ in A, its component R_0 is also equal to 0. Then Step 2 gives $0 = x_1 \otimes R_0 = x_1 \otimes \sum_{i=1}^m m x_1^{m-1} Q_m$ where Q_i are polynomials defined in Step 1. Then all $Q_i = 0$. ■

Algebras with filtration

REMARK: Step 3 of the proof of previous lemma uses $\operatorname{char} k = 0$. **Hopf theorem is false for $\operatorname{char} k > 0$.**

DEFINITION: Filtration on an algebra A is a sequence of subspaces $A_0 \supset A_1 \supset A_2 \supset ...$ such that $A_i \cdot A_i \subset A_{i+j}$

EXAMPLE: Let $I \subset A$ be an ideal. **the I-adic filtration** is the filtration by the degrees of the ideal I: $A \supset I \supset I^2 \supset I^3 \supset ....$

DEFINITION: Let $A_0 \supset A_1 \supset A_2 \supset ...$ be a filtered algebra. **The associated graded algebra** is $A_{gr} := \bigoplus_i A_i/A_{i+1}$.

LEMMA: Let $A \supset I \supset I^2 \supset I^3 \supset ...$ be an adic filtration, and $A_{gr} := \bigoplus_i I^i/I^{i+1}$ the associated graded algebra. **Then A_{gr} is generated by its first and second graded components $A/I \oplus I/I^2$.**

Proof: Indeed, I^k/I^{k+1} is generated by products of k elements in (I/I^2). ■

The augmentation ideal

DEFINITION: Augmentation ideal in a bialgebra is the kernel of the counit homomorphism $\varepsilon : A \longrightarrow k$.

CLAIM: Let $Z \subset A$ be the augmentation ideal. Then

$$\Delta(x) = 1 \otimes x + x \otimes 1 \mod (Z \otimes Z). \quad (**)$$

for any $x \in Z$.

Proof: Indeed, $Z \otimes Z = \ker(\varepsilon \otimes \mathrm{Id}_A) \cap \ker(\mathrm{Id}_A \otimes \varepsilon)$. The counit condition gives $x = [\varepsilon \otimes \mathrm{Id}_A](\Delta(x))$ and $x = [\mathrm{Id}_A \otimes \varepsilon](\Delta(x))$, while

$$[\varepsilon \otimes \mathrm{Id}_A](1 \otimes x + x \otimes 1) = \varepsilon(x) + x = [\mathrm{Id}_A \otimes \varepsilon](1 \otimes x + x \otimes 1).$$

Comparing these equations, we obtain

$$\Delta(x) - 1 \otimes x - x \otimes 1 \in Z \otimes Z.$$

when $\varepsilon(x) = 0$. ■

Proof of Hopf theorem

THEOREM: (Hopf theorem) Let A be a graded bialgebra of finite type over a field k of characteristic 0. **Then A is a free graded commutative k-algebra**.

Proof. Step 1: Consider the filtration of A by the degrees of the augmentation ideal Z, and let $A_{gr} := \bigoplus_i Z^i/Z^{i+1}$ be the associated graded algebra. **Since $\Delta(Z) = 1\otimes x + x\otimes 1 \mod (Z\otimes Z)$, one has $\Delta(Z^p) \subset \bigoplus_{i+j=p} Z^i \otimes Z^j$.**

Step 2: Consider the filtration on $A\otimes A$ by the powers of the ideal $\tilde Z := Z\otimes 1 + 1\otimes Z$. The natural map

$$\bigoplus_{p+q=n} \frac{Z^p}{Z^{p+1}} \otimes \frac{Z^q}{Z^{q+1}} \longrightarrow \bigoplus_n \frac{\tilde Z^n}{\tilde Z^{n+1}}$$

is by construction surjective, and takes graded components to graded components of the same dimension, hence $A_{gr}\otimes A_{gr}$ is isomorphic to $\bigoplus_n \frac{\tilde Z^n}{\tilde Z^{n+1}}$ (these components are finite dimension because A^* is of finite type). Step 1 implies that $\Delta(Z^n) \subset \tilde Z^n$, hence **the comultiplication $\Delta: A_{gr} \longrightarrow A_{gr}\otimes A_{gr}$ induces a bialgebra structure on A_{gr}.**

Proof of Hopf theorem (2)

Step 3: The algebra A_{gr} is multiplicatively generated by Z^1/Z^2. Since $\Delta(x) = 1 \otimes x + x \otimes 1 \mod (Z \otimes Z)$, all elements of Z^1/Z^2 are primitive in A_{gr}. Therefore, the algebra A_{gr} is generated by primitive elements. This implies that A_{gr} **is a free algebra generated by its space of primitive elements.**

Step 4: Let x_i be a basis in the space of primitive elements of A_{gr}, and let $\tilde{x}_i$ be a representative of each of $x_i \in Z^k/Z^{k+1}$ in Z_k, of the same parity as x_i. **Since there is no non-trivial relations between x_i, there are no non-trivial relations between $\tilde{x}_i$.** It remains to show that $\tilde{x}_i$ generate A.

Step 5: Return to the grading originally given on A. Since ε is compatible with grading, the ideal Z is a direct sum of its graded components, and the algebra A_{gr} is equipped with a grading induced from A. Dimensions of the graded components A^p and A^p_{gr} of A and A_{gr} are equal, because any filtered space is isomorphic as a vector space to its associated graded space. Let $\{y_i\}$ be a set of monomials of $x_i \in A_{br}$ giving a basis in the graded component A^p_{gr}, and $\{\tilde{y}_i\}$ the corresponding monomials in A^p. **Since $\{y_i\}$ are linearly independent, the monomials $\{\tilde{y}_i\}$ are linearly independent, and since $\dim A^p = \dim A^p_{gr}$, these monomials generate A^p.** We have shown that $A^\bullet$ is freely generated by the vectors $\{\tilde{y}_i\}$. ■

Grassmannian $\mathrm{Gr}(n) := \mathrm{Gr}(n, \infty)$ as a classifying space (reminder)

DEFINITION: Consider the natural embeddings

$$\mathrm{Gr}(n, m) \hookrightarrow \mathrm{Gr}(n, m+1) \hookrightarrow \mathrm{Gr}(n, m+2) \hookrightarrow ...,$$

associated with the maps $\mathbb{K}^m \hookrightarrow \mathbb{K}^{m+1} \hookrightarrow \mathbb{K}^{m+2} \hookrightarrow$ ... and **let $\mathrm{Gr}(n) := \mathrm{Gr}(n, \infty)$ denote the union of all these spaces** (that is, the inductive limit of cellular complexes). By definition, $\mathrm{Gr}(n)$ is the space of n-dimensional subspaces in $\mathbb{K}^\infty$, where $\mathbb{K}^\infty$ denotes. $\bigoplus_{i=0}^\infty \mathbb{K}$.

DEFINITION: The fundamental bundle on $\mathrm{Gr}(n)$ is the vector bundle with fiber W at a point $W \subset \mathbb{K}^\infty$.

THEOREM: Let B be a vector bundle of rank n on a cellular space X. Then there exists a continuous map $\varphi : X \longrightarrow \mathrm{Gr}(n)$ such that **B is isomorphic to the pullback $\varphi^* B_{\mathrm{fun}}$ of the fundamental bundle.**

Proof: Last lecture. ∎

REMARK: In fact, **$\mathrm{Gr}(n)$ is the classifying space of vector bundles of rank n**, in the sense that **isomorphism classes of vector bundles on X are in bijective correspondence with homotopy classes of maps $\varphi : X \longrightarrow \mathrm{Gr}(n)$.**

The infinite Grassmannian (reminder)

DEFINITION: Choose a basis $x_0, x_1, ...$, in $\mathbb{C}^\infty$ or $\mathbb{R}^\infty$, and let $R: \mathbb{C}^\infty \longrightarrow \mathbb{C}^\infty$ be defined as above, $R(x_i) = x_{i+1}$. Consider the embedding $Gr(n,\infty) \hookrightarrow Gr(n+1,\infty)$, taking the space $L \subset \mathbb{C}^\infty$ to $\langle x_0, R(L)\rangle$. The union (inductive limit) $\bigcup_n Gr(n,\infty)$ is called **the infinite Grassmannian**, and is denoted as BU.

DEFINITION: Vector bundles B_1, B_2 are called **stably equivalent** if $B_1 \oplus U_1 \cong B_2 \oplus U_2$, where U_i are trivial vector bundles.

THEOREM: Let X be a finite cellular space. Then **homotopy classes of maps $X \longrightarrow BG$ are in bijective correspondence with classes of stable equivalence of vector bundles.**

Proof: Left as an exercise. ∎

BU as an H-space (reminder)

Bott periodicity identifies the space of loops on U and BU; this implies that BU is an H-space (loop spaces are H-spaces). However, the H-structure on BU can be constructed explicitly.

PROPOSITION: Consider a map $S: \mathbb{C}^\infty \times \mathbb{C}^\infty \longrightarrow \mathbb{C}^\infty$ taking the basis vectors x_i of the first space to x_{2i} and the basis vectors of the second space to x_{2i+1}. Then $L, L' \longrightarrow S(L, L')$ **defines a structure of an H-space on the infinite Grassmannian BU.**

Proof: We need to show H-associativity, which would follow if we prove that any permutation of basis elements is homotopic to identity. This is left as an exercise. ■

COROLLARY: **$H^*(BU, \mathbb{Q})$ is a free supercommutative algebra.**

Proof: Follows from Hopf theorem. ■

REMARK: It is not hard to write a celular decomposition for the Grassmannian ("Schubert cells"); all cells are even-dimensional, which gives the dimensions of the groups $H^i(BG)$. Then the Hopf theorem can be used to compute the cohomology algebra. This also implies that **the algebra $H^*(BG)$ is commutative (and therefore, free polynomial).**

Cohomology of BU (reminder)

CLAIM: $H^*(BU, \mathbb{Q})$ **is a free polynomial algebra generated by classes** $c_1, c_2, ...$ **in all even degrees.**

Proof: Consider the principal U-bundle $E \longrightarrow BU$. Cohomology of U is a free graded commutative algebra with one generator in each odd degree, as shown earlier today.

$E_2^{p,q}$-term of the Leray-Serre spectral sequence is $H^p(BU) \otimes H^q(U)$. Since this sequence converges to 0, every generator of $H^*(U)$ has to go to a generator of $H^*(BU)$.

For each generator of $H^*(U)$ of degree $2n+1$, the corresponding generator has degree $2n+2$, **hence there is one generator in each even degree.** ■

Chern classes: axiomatic definition

DEFINITION: Chern classes are classes $c_i(B) \in H^{2i}(X)$, $i = 0, 1, 2, ...$, defined for each complex vector bundle B on a cellular space X and satisfying the following acioms.

1. $c_0(B) = 1$.

2. **functoriality:** for each continuous map $f: X \longrightarrow Y$ we have $f^* c_i(B) = c_i(f^* B)$.

3. **Whitney formula:** $c_*(B \oplus B') = c_*(B)c_*(B')$, where $c_*(B) = \sum_i c_i(B)$ **("full Chern class")**.

4. **normalization:** Let $\mathcal{O}(i)$ be the standard bundle on a complex projective space. Then $c_1(\mathcal{O}(1)) = [H]$, where $[H]$ is the fundamental class of a hyperplane section. For all $i > i$, we have $c_i(\mathcal{O}(1)) = 0$.

REMARK: From functoriality it follows that $c_i(B) = 0$ when B is trivial and $i > 0$.

Chern classes on $BU(1)^n$

To prove the uniqueness of Chern classes, we start with the following exercise.

EXERCISE: Prove that $BU(1) = \mathbb{C}P^\infty$.

DEFINITION: The fundamental bundle on $BU(1)^n$ is a rank n bundle on $BU(1)^n = (\mathbb{C}P^\infty)^n$ obtained by taking a direct sum of the fundamental bundles pulled back from each $BU(1)$.

REMARK: The Chern classes of the fundamental bundle on $BU(1)^n$ ate uniquely determined from the axioms. Indeed, the fundamental bundle on $BU(1) = \mathbb{C}P^\infty$ is $\mathcal{O}(1)$, its Chern class is the generator of $H^*(\mathbb{C}P^\infty, \mathbb{Q}) = \mathbb{Q}([H])$, and the Chern classes of a direct sum are determined from the Whitney formula.

THEOREM: (Splitting principle): Let $\varphi_{\text{fun}} : BU(1)^n \longrightarrow BU$ be the map associated with the fundamental bundle composed with the embedding $\operatorname{Gr}(n) \longrightarrow BU$. **Then φ_{fun} induces an injective map on cohomology up to degree n, and takes the primitive generator $\chi_d \in H^{2d}(BU)$ to the class $\lambda \sum_{i=1}^n z_i^d$ in the polynomial algebra $H^*(BU(1)^n) = \mathbb{Q}[z_1, ..., z_n]$.**

Proof: Next lecture, **if there is demand.** ■

Chern classes: uniqueness

THEOREM: The Chern classes are uniquely determined from the axioms.

Proof. Step 1: Every bundle is a pullback of the fundamental bundle on $BU(n) = \operatorname{Gr}(n, \infty)$. Therefore, the Chern classes are obtained as a pullbacks of the Chern classes of the fundamental bundle on $BU(n)$. Since $c_i(B) = c_i(B \oplus \text{trivial bundle})$, **these classes are restrictions of classes in $H^*(BU)$.**

Step 2: Consider the map $BU(1)^n \longrightarrow BU(n)$ induced by the fundamental bundle. This map is injective on cohomology, as follows from the splitting principle. Since the Chern classes of the fundamental bundle on $BU(1)^n$ are determined from the axioms, and $H^*(BU(n)) \subset H^*(BU(1)^n)$, **the Chern classes of the fundamental bundle on $BU(n)$ are determined from the axioms.** ■

REMARK: We just **proved uniqueness of Chern classes satisfying the axiomatic definition.** The easiest way to show existence of Chern classes satisfying the axioms is **to prove the Hopf theorem and use its proof.**

Primitive generators of $H^*(BU)$

REMARK: Any class $a \in H^i(BU)$ can be evaluated on a bundle B over X, producing a class $a(B) \in H^i(X)$. Indeed, B is the pullback of the fundamental bundle B_{fun} on $\operatorname{Gr}(n)$, giving a map $\varphi_B : X \longrightarrow BU$, and we set $a(B) = \varphi_B^*(a)$.

REMARK: Historically, this was done by using invariant polynomials on curvature, for some Hermitian connection on B.

REMARK: Consider a map $\varphi = (\varphi_1, \varphi_2) : X \longrightarrow BU \times BU$, associated to bundles B_1, B_2. Composition of φ and the H-multiplication map $\mu : BU \times BU \longrightarrow BU$ produces a map $\varphi \circ \mu : X \longrightarrow BU$ associated with the bundle $B_1 \oplus B_2$. Therefore, the comultiplication map $\Delta : H^*(BU) \longrightarrow H^*(BU) \otimes H^*(BU)$ takes $\varphi^* : H^*(BU) \otimes H^*(BU) \longrightarrow H^*(X)$ to $\Delta \circ \varphi^* : H^*(BU) \longrightarrow H^*(X)$ **mapping a class in $\operatorname{Map}(X, BU \times BU)$ associated with the pair B_1, B_2, to the class in $\operatorname{Map}(X, BU)$ associated with $B_1 \oplus B_2$.**

COROLLARY: Let $x \in H^*(BU)$. **Then $x(B_1 \oplus B_2) = \Delta(x)(B_1, B_2)$.** ■

COROLLARY: Let $x \in H^*(BU)$ be a primitive class. **Then $x(B_1 \oplus B_2) = x(B_1) + x(B_2)$.**

Proof: Since $\Delta(x) = x \otimes 1 + 1 \otimes x$, the class $\Delta(x)$ evaluated on B_1, B_2 is equal to $x(B_1) + x(B_2)$. ■

Classes satisfying the Whitney formula

REMARK: We will construct the full Chern class as $c_*(B)$, where $c_* \in H^*(BU)$ is a certain cohomology class.

REMARK: Let $\chi_i \in H^{2i}(BU)$ be a primitive generator; by Hopf theorem, this class is unique up to a constant. Since $\chi_i(B_1 \oplus B_2) = \chi_i(B_1) + \chi_i(B_2)$, **the class $C := e^{\sum_i a_i \chi_i}$ satisfies the Whitney formula $C(B_1 \oplus B_2) = C(B_1)C(B_2)$, for any collection of coefficients $a_i \in \mathbb{Q}$.** To construct Chern classes satisfying the axioms above, **it remains to arrange the coefficients a_i in such a way that $C(\mathcal{O}(1)) = 1 + [H]$.**

LEMMA: Consider the natural map $\varphi: BU(1) \longrightarrow BU$ associated with the fundamental bundle on $BU(1)$. **Then $\varphi^*(\chi_i) \neq 0$.**

Proof: Follows immediately from the splitting principle, because $\varphi^*(\chi_i) = \lambda \sum_j z_j^n$, and the map $BU(1) \longrightarrow BU(1)^n$ takes $\sum_j z_j^n \in H^*(BU(1)^n)$ to $z_1^n \in H^*(BU(1))$. ■

Construction of Chern classes

Existence of Chern classes is given by the following easy corollary.

THEOREM: Let $\varphi: BU(1) \longrightarrow BU$ be the standard map associated with the fundamental bundle. Choose the generators $\chi_i \in H^{2i}(BU)$ in such a way that $\varphi^*(\chi_i) = \log(1+[H])$, and let $c_*(B) := \exp\left(\sum_i \chi_i\right)$. **Then $c_*(\mathcal{O}(1)) = 1+[H]$ and $c_*(B_1 \oplus B_2) = c_*(B_1)c_*(B_2)$.**

Proof: $c_*(\mathcal{O}(1)) = 1+[H]$ because $\varphi^*(\chi_i) = \log(1+[H])$, hence $\exp\left(\varphi^*(\sum_i \chi_i)\right) = 1+[H]$. Whitney formula $c_*(B_1 \oplus B_2) = c_*(B_1)c_*(B_2)$ is true when c_* is an exponent of any sum of primitive elements. ∎