K3 surfaces

lecture 4: Chern classes

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Grassmannian $\operatorname{Gr}(n) := \operatorname{Gr}(n,\infty)$ as a classifying space (reminder)

DEFINITION: Consider the natural embeddings

$$\operatorname{Gr}(n,m) \hookrightarrow \operatorname{Gr}(n,m+1) \hookrightarrow \operatorname{Gr}(n,m+2) \hookrightarrow ...,$$

associated with the maps $\mathbb{K}^m \hookrightarrow \mathbb{K}^{m+1} \hookrightarrow \mathbb{K}^{m+2} \hookrightarrow$ ... and **let** $\operatorname{Gr}(n) := \operatorname{Gr}(n,\infty)$ **denote the union of all these spaces** (that is, the inductive limit of cellular complexes). By definition, $\operatorname{Gr}(n)$ is the space of n-dimensional subspaces in $\mathbb{K}^\infty$, where $\mathbb{K}^\infty$ denotes. $\bigoplus_{i=0}^\infty \mathbb{K}$.

DEFINITION: **The fundamental bundle** on $\operatorname{Gr}(n)$ is the vector bundle with fiber W at a point $W \subset \mathbb{K}^\infty$.

THEOREM: Let B be a vector bundle of rank n on a cellular space X. Then there exists a continuous map $\varphi: X \longrightarrow \operatorname{Gr}(n)$ such that B **is isomorphic to the pullback** $\varphi^* B_{\text{fun}}$ **of the fundamental bundle.**
Proof: Last lecture. ■

REMARK: In fact, $\operatorname{Gr}(n)$ **is the classifying space of vector bundles of rank** n, in the sense that **isomorphism classes of vector bundles on** X **are in bijective correspondence with homotopy classes of maps** $\varphi: X \longrightarrow \operatorname{Gr}(n)$.

The infinite Grassmannian (reminder)

DEFINITION: Choose a basis $x_0, x_1, ...$, in $\mathbb{C}^\infty$ or $\mathbb{R}^\infty$, and let $R: \mathbb{C}^\infty \longrightarrow \mathbb{C}^\infty$ be defined as above, $R(x_i) = x_{i+1}$. Consider the embedding $Gr(n, \infty) \hookrightarrow Gr(n+1, \infty)$, taking the space $L \subset \mathbb{C}^\infty$ to $\langle x_0, R(L)\rangle$. The union (inductive limit) $\bigcup_n Gr(n, \infty)$ is called **the infinite Grassmannian**, and is denoted as BU.

DEFINITION: Vector bundles B_1, B_2 are called **stably equivalent** if $B_1 \oplus U_1 \cong B_2 \oplus U_2$, where U_i are trivial vector bundles.

THEOREM: Let X be a finite cellular space. Then **homotopy classes of maps $X \longrightarrow BU$ are in bijective correspondence with classes of stable equivalence of vector bundles.**

Proof: Left as an exercise. ■

BU as an H-space (reminder)

Bott periodicity identifies the space of loops on U and BU; this implies that BU is an H-space (loop spaces are H-spaces). However, the H-structure on BU can be constructed explicitly.

PROPOSITION: Consider a map $S: \mathbb{C}^\infty \times \mathbb{C}^\infty \longrightarrow \mathbb{C}^\infty$ taking the basis vectors x_i of the first space to x_{2i} and the basis vectors of the second space to x_{2i+1}. Then $L, L' \longrightarrow S(L, L')$ **defines a structure of an H-space on the infinite Grassmannian BU.**

Proof: We need to show H-associativity, which would follow if we prove that any permutation of basis elements is homotopic to identity. This is left as an exercise. ■

COROLLARY: **$H^*(BU, \mathbb{Q})$ is a free supercommutative algebra.**

Proof: Follows from Hopf theorem. ■

REMARK: It is not hard to write a celular decomposition for the Grassmannian ("Schubert cells"); all cells are even-dimensional, which gives the dimensions of the groups $H^i(BU)$. Then the Hopf theorem can be used to compute the cohomology algebra. This also implies that **the algebra $H^*(BU)$ is commutative (and therefore, free polynomial).**

Cohomology of BU (reminder)

CLAIM: $H^*(BU,\mathbb{Q})$ **is a free polynomial algebra generated by classes** $c_1, c_2, ...$ **in all even degrees.**

Proof: Consider the principal U-bundle $E \longrightarrow BU$. Cohomology of U is a free graded commutative algebra with one generator in each odd degree, as shown earlier today.

$E_2^{p,q}$-term of the Leray-Serre spectral sequence is $H^p(BU)\otimes H^q(U)$. Since this sequence converges to 0, every generator of $H^*(U)$ has to go to a generator of $H^*(BU)$.

For each generator of $H^*(U)$ of degree $2n+1$, the corresponding generator has degree $2n+2$, **hence there is one generator in each even degree.** ■

Chern classes: axiomatic definition

DEFINITION: Chern classes are classes $c_i(B) \in H^{2i}(X)$, $i = 0, 1, 2, ...$, defined for each complex vector bundle B on a cellular space X and satisfying the following acioms.

1. $c_0(B) = 1$.

2. **functoriality:** for each continuous map $f: X \longrightarrow Y$ we have $f^* c_i(B) = c_i(f^* B)$.

3. **Whitney formula:** $c_*(B \oplus B') = c_*(B) c_*(B')$, where $c_*(B) = \sum_i c_i(B)$ **("full Chern class")**.

4. **normalization:** Let $\mathcal{O}(i)$ be the standard bundle on a complex projective space. Then $c_1(\mathcal{O}(1)) = [H]$, where $[H]$ is the fundamental class of a hyperplane section. For all $i > i$, we have $c_i(\mathcal{O}(1)) = 0$.

REMARK: From functoriality it follows that $c_i(B) = 0$ when B is trivial and $i > 0$.

The splitting principle

EXERCISE: Prove that $BU(1) = \mathbb{C}P^\infty$.

DEFINITION: The fundamental bundle on $BU(1)^n$ is a rank n bundle on $BU(1)^n = (\mathbb{C}P^\infty)^n$ obtained by taking a direct sum of the fundamental bundles pulled back from each $BU(1)$.

REMARK: The Chern classes of the fundamental bundle on $BU(1)^n$ ate uniquely determined from the axioms. Indeed, the fundamental bundle on $BU(1) = \mathbb{C}P^\infty$ is $\mathcal{O}(1)$, its Chern class is the generator of $H^*(\mathbb{C}P^\infty, \mathbb{Q}) = \mathbb{Q}([H])$, and the Chern classes of a direct sum are determined from the Whitney formula.

REMARK: The cohomology of $BU(1)$ is the polynomial algebra, generated by $[H] \in H^2(BU(1))$. By Künnethe formula, $H^*(BU(1)^n) = \mathbb{Z}[z_1, ..., z_n]$, where $z_i \in H^2(BU(1))$ is the generator in the i-th component of the product.

THEOREM: (Splitting principle): Let $\varphi_{\text{fun}} : BU(1)^n \longrightarrow BU$ be the map associated with the fundamental bundle composed with the embedding $\operatorname{Gr}(n) \longrightarrow BU$. **Then φ_{fun} induces an injective map in cohomology up to degree n.** Moreover, the pullback of a primitive generator in $H^{2k}(BU)$ is proportional to $\sum_{i=1}^n z_i^k$, if $z_i \in H^2(BU(1)^n)$ is defined as above.

Proof: Home assignment 3. ■

Chern classes: uniqueness

THEOREM: The Chern classes are uniquely determined from the axioms.

Proof. Step 1: Every bundle is a pullback of the fundamental bundle on $BU(n) = \operatorname{Gr}(n, \infty)$. Therefore, the Chern classes of B are obtained as a pullbacks of the Chern classes of the fundamental bundle on $BU(n)$. Since $c_i(B) = c_i(B \oplus \text{trivial bundle})$, **these classes are restrictions of classes in $H^*(BU)$.**

Step 2: Consider the map $BU(1)^n \longrightarrow BU(n)$ induced by the fundamental bundle. This map is injective on cohomology, as follows from the splitting principle. Since the Chern classes of the fundamental bundle on $BU(1)^n$ are determined from the axioms, and $H^*(BU(n)) \subset H^*(BU(1)^n)$, **the Chern classes of the fundamental bundle on $BU(n)$ are determined from the axioms.** ■

REMARK: We just **proved uniqueness of Chern classes satisfying the axiomatic definition.** The easiest way to show existence of Chern classes satisfying the axioms is **to use the primitive generators of cohomology of BU obtained from the Hopf theorem.**

Primitive generators of $H^*(BU)$

REMARK: Any class $a \in H^i(BU)$ can be evaluated on a bundle B over X, producing a class $a(B) \in H^i(X)$. Indeed, B is the pullback of the fundamental bundle B_{fun} on $\operatorname{Gr}(n)$, giving a map $\varphi_B : X \longrightarrow BU$, and we set $a(B) = \varphi_B^*(a)$.

REMARK: Historically, this was done by using invariant polynomials on curvature, for some Hermitian connection on B.

REMARK: Consider a map $\varphi = (\varphi_1, \varphi_2) : X \longrightarrow BU \times BU$, associated to bundles B_1, B_2. Composition of φ and the H-multiplication map $\mu : BU \times BU \longrightarrow BU$ produces a map $\varphi \circ \mu : X \longrightarrow BU$ associated with the bundle $B_1 \oplus B_2$. Therefore, the comultiplication map $\Delta : H^*(BU) \longrightarrow H^*(BU) \otimes H^*(BU)$ takes $\varphi^* : H^*(BU) \otimes H^*(BU) \longrightarrow H^*(X)$ to $\Delta \circ \varphi^* : H^*(BU) \longrightarrow H^*(X)$ **mapping a class in $\operatorname{Map}(X, BU \times BU)$ associated with the pair B_1, B_2, to the class in $\operatorname{Map}(X, BU)$ associated with $B_1 \oplus B_2$.**

COROLLARY: Let $x \in H^*(BU)$. **Then $x(B_1 \oplus B_2) = \Delta(x)(B_1, B_2)$.** ■

COROLLARY: Let $x \in H^*(BU)$ be a primitive class. **Then $x(B_1 \oplus B_2) = x(B_1) + x(B_2)$.**

Proof: Since $\Delta(x) = x \otimes 1 + 1 \otimes x$, the class $\Delta(x)$ evaluated on B_1, B_2 is equal to $x(B_1) + x(B_2)$. ■

Classes satisfying the Whitney formula

REMARK: We will construct the full Chern class $c_*(B)$ **as a pullback of a class $C \in H^*(BU)$.**

REMARK: Let $Ch_i \in H^{2i}(BU)$ be a primitive generator; by Hopf theorem, this class is unique up to a constant. Since $Ch_i(B_1 \oplus B_2) = Ch_i(B_1) + Ch_i(B_2)$, **the class $C := e^{\sum_i a_i Ch_i}$ satisfies the Whitney formula $C(B_1 \oplus B_2) = C(B_1)C(B_2)$, for any collection of coefficients $a_i \in \mathbb{Q}$.** To construct Chern classes satisfying the axioms above, **it remains to arrange the coefficients a_i in such a way that $C(\mathcal{O}(1)) = 1 + [H]$.**

LEMMA: Consider the natural map $\varphi : BU(1) \longrightarrow BU$ associated with the fundamental bundle on $BU(1)$. **Then $\varphi^*(Ch_i) \neq 0$.**

Proof: Follows immediately from the splitting principle, because $\varphi^*(Ch_i) = \lambda \sum_j z_j^n$, and the map $BU(1) \longrightarrow BU(1)^n$ takes $\sum_j z_j^n \in H^*(BU(1)^n)$ to $z_1^n \in H^*(BU(1))$. ■

Construction of Chern classes

Existence of Chern classes is given by the following easy corollary.

THEOREM: Let $\varphi: BU(1) \longrightarrow BU$ be the standard map associated with the fundamental bundle. Choose the generators $Ch_i \in H^{2i}(BU)$ in such a way that $\varphi^*\left(\sum_i Ch_i\right) = \log(1+[H])$, and let $c_*(B) := \exp\left(\sum_i Ch_i\right)$. **Then $c_*(\mathcal{O}(1)) = 1+[H]$ and $c_*(B_1 \oplus B_2) = c_*(B_1)c_*(B_2)$.**

Proof: $c_*(\mathcal{O}(1)) = 1+[H]$ because $\varphi^*(Ch_i) = \log(1+[H])$, hence $\exp\left(\varphi^*(\sum_i Ch_i)\right) = 1+[H]$. Whitney formula $c_*(B_1 \oplus B_2) = c_*(B_1)c_*(B_2)$ is true when c_* is an exponent of any sum of primitive elements. ■