K3 surfaces

lecture 5: Riemann-Roch-Hirzebruch formula

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Chern classes (reminder)

DEFINITION: Chern classes are classes $c_i(B) \in H^{2i}(X)$, $i = 0, 1, 2, ...$, defined for each complex vector bundle B on a cellular space X and satisfying the following acioms.

1. $c_0(B) = 1$.

2. **functoriality:** for each continuous map $f: X \longrightarrow Y$ we have $f^* c_i(B) = c_i(f^* B)$.

3. **Whitney formula:** $c_*(B \oplus B') = c_*(B) c_*(B')$, where $c_*(B) = \sum_i c_i(B)$ **("full Chern class")**.

4. **normalization:** Let $\mathcal{O}(i)$ be the standard bundle on a complex projective space. Then $c_1(\mathcal{O}(1)) = [H]$, where $[H]$ is the fundamental class of a hyperplane section. For all $i > i$, we have $c_i(\mathcal{O}(1)) = 0$.

REMARK: From functoriality it follows that $c_i(B) = 0$ when B is trivial and $i > 0$.

THEOREM: Chern classes exist and are determined by these axioms.

The K-group

DEFINITION: Let M be a topological space, and V the abelian group, freely generated by the isomorphic classes of vector bundles on M. The **the K-group** is the quotient of V by its subgroup generated by relations $[F_1]+[F_3]=[F_2]$ for all exact sequences of $0\longrightarrow F_1\longrightarrow F_2\longrightarrow F_3\longrightarrow 0$.

REMARK: The H-group structure on BU defines a group structure on the homotopy classes of maps $X\longrightarrow BU$. **By construction of the H-structure, $[\varphi_1]+[\varphi_3]=[\varphi_2]$,** where $\varphi_i:\ X\longrightarrow BU$ is the map associated with the bundle F_i on X, and $[\varphi_i]$ their homotopy classes. This implies the claim

CLAIM: K-group of X is naturally isomorphic with the set of homotopy classes of maps $\varphi:\ X\longrightarrow BU$, **equipped with a group structure induced by the H-group structure on BU.** ■

COROLLARY: Chern classes are well defined on elements of the K-group, and satisfy the Whitney formula.

Proof: Let $C_*\in H^*(BU)$ be the total Chern class of the fundamental bundle on BU. **Then $c_*(F):=\varphi^*(C_*)$, for any F in K-group associated with $\varphi:\ X\longrightarrow BU$.** For Chern classes defined this way for the maps $\varphi:\ X\longrightarrow BU$ the Whitney relations are already proven (Lecture 4). ■

Coherent sheaves

DEFINITION: Let M be a complex manifold, and $\mathcal{O}_M$ its structure sheav (the sheaf of holomorphic functions). **Coherent sheaf** is a sheaf of $\mathcal{O}_M$-modules, locally isomorphic to a quotient of a free sheaf $\mathcal{O}_M^n$ by a finitely generated $\mathcal{O}_M$-invariant subsheaf.

REMARK: In algebraic category, the definition is the same, but "locally" means "locally in Zariski topology". Serre's GAGA principle implies that **on a projective manifolds these two definitions are equivalent.**

EXERCISE: Let M be a projective manifold. **Prove that any coherent sheaf F has a resolution $0 \longrightarrow B_n \longrightarrow B_{n-1} \longrightarrow ... \longrightarrow B_0 \longrightarrow F \longrightarrow 0$, where B_i are vector bundles.**

DEFINITION: Such a resolution is called **a syzygy** of the sheaf F. Clearly, $[F] = \sum_i (-1)^i [B_i]$ in the K-group of coherent sheaves.

Coherent sheaves and their Chern classes

REMARK: After this is done, it is possible to prove that **the K-group of coherent sheaves on a projective manifold is equal to the K-group generated by holomorphic vector bundles.**

DEFINITION: The **Chern class** of a coherent sheaf is the Chern class of the corresponding element of the K-group.

REMARK: In complex analytic category, **we have to blow up the manifold to resolve the singularities of the coherent sheaf, take the Chern class of its resolution, and apply the pushforward in cohomology.** However, for most practical purposes, this is not necessary, because the syzygy can be constructed explicitly.

Euler characteristic of a coherent sheaf

DEFINITION: Let F be a coherent sheaf. Its **Euler characteristic** $\chi(F)$ is the number $\chi(F) := \sum_i (-1)^i \dim H^i(F)$..

CLAIM: For any exact sequence $0 \longrightarrow F_1 \longrightarrow F_2 \longrightarrow F_3 \longrightarrow 0$ of coherent sheaves, **we have $\chi(F_2) = \chi(F_1) + \chi(F_3)$.**

Proof: Left as an exercise. ■

REMARK: This implies that χ **defines a homomorphism** $K(M) \xrightarrow{\chi} \mathbb{Z}$, where $K(M)$ denotes the K-group.

Chern character

Let $\varphi: BU(1) \longrightarrow BU$ be the standard map, and $Ch_i \in H^{2i}(BU)$ the primitive generators of $H^*(BU, \mathbb{Z})$ defined in such a way that $c_1(\mathcal{O}(1)) = \varphi^*\left(\sum_i Ch_i\right) = \log(1 + [H])$. Since Ch_i are primitive, **the map $\varphi \mapsto \varphi^*(Ch_i)$ defines a homomorphism from $K(X)$ to $H^*(X)$,** where φ is a map $X \to BU$ interpreted as an element of the K-group.

DEFINITION: Let X be a CW-space. **Chern character** $ch_*: K(X) \longrightarrow H^*(X, \mathbb{Q})$ is a map which associates to each map $\varphi: X \longrightarrow BU$ the class $\varphi^*(Ch_*) \in H^*(X, \mathbb{Q})$, where $Ch_* \in H^*(BU, \mathbb{Q})$ is the cohomology class defined above.

REMARK: An alternative way to define the Chern character is to **define it on line bundles as $ch_*(L) := \exp(c_1(L))$, and extend to a homomorphism from the K-group using the splitting principle.**

REMARK: The K-group is in fact a ring; the ring structure is defined on its generators (the vector bundles) by taking the tensor product.

EXERCISE: Using the splitting principle, prove that **the Chern character $ch_*: K(X) \longrightarrow H^*(X, \mathbb{Q})$ is a ring homomorphism.**

Riemann-Roch-Hirzebruch theorem

THEOREM: (Riemann-Roch-Hirzebruch)
Let F be a coherent sheaf on a compact complex manifold M. **Then $\chi(F)$ can be expressed through the Chern classes of F and M as follows:**

$$\chi(F) = \int_X ch_*(F) \wedge td_*(TM),$$

wheere $td_*(TM)$ denotes **the total Todd class of the tangent bundle TM**,

$$td_* = 1 + \frac{c_1}{2} + \frac{c_1^2 + c_2}{12} + \frac{c_1 c_2}{24} + \frac{-c_1^4 + 4c_1^2 c_2 + c_1 c_3 + 3c_2^2 - c_4}{720} + ...$$

REMARK: Formally, **the Todd class can be defined using the splitting principle** as follows: it is a polynomial on Chern classes which satisfies the Whitney formula $td_*(B \oplus B') = td_*(B) td_*(B')$, and for a line bundle L with $c_1(L) = \alpha$, we have $td_*(L) = \frac{\alpha}{1-e^{-\alpha}}$.

REMARK: We will prove the Riemann-Roch-Hirzebruch formula for several special cases, and use it only in these cases; **a general expression is here for your enlightenment only.**

K-group for complex curves

LEMMA: Let M be a smooth compact complex curve. Then **the K group of M is generated by line bundles.**

Proof. Step 1: Using Kodaira vanishing theorem, we obtain that there exists a non-zero map from a line bundle to any coherent sheaf F of positive rank. Indeed, $F \otimes L^N$ has a section, giving a non-zero monomorphism $\mathcal{O} \hookrightarrow F \otimes L^N$, or, equivalently, $L^{-N} \hookrightarrow F$.

Step 2: This gives an exact sequence $0 \longrightarrow F_1 \longrightarrow F \longrightarrow F_2 \longrightarrow 0$, with $\operatorname{rk} F_1 < \operatorname{rk} F$, $\operatorname{rk} F_2 < \operatorname{rk} F$. **Using induction on rank, we reduce the main statement to sheaves of rank 1 and 0.**

Step 3: Also, a sheaf of rank 0 is a direct sum of sheaves F_i fitting into the exact sequence $0 \longrightarrow I \longrightarrow \mathcal{O} \longrightarrow F_i \longrightarrow 0$, where $I \subset \mathcal{O}$ is an ideal sheaf. **(prove this)**. This implies that rank 0 coherent sheaves also belong to the subgroup of the K group generated by line bundles. ■

Riemann-Roch theorem for complex curves

THEOREM: (Riemann-Roch theorem for complex curves)
Let F be a coherent sheaf on a compact complex curve of genus g. **Then**

$$\chi(F) = c_1(F) + \operatorname{rk}(F)(1-g). \qquad (*)$$

Proof. Step 1: Both sides of the equation (*) define a homomorphism from $K(M)$ to $\mathbb{Z}$. Therefore, it suffices to check (*) on any set of generators of $K(M)$. By the previous lemma, **it is sufficient to prove that (*) holds for line bundles.**

Step 2: For $F = \mathcal{O}$ the statement is clear. When $\operatorname{rk} F = 0$, we obtain F as extension of sheaves of form $k_x := \mathcal{O}/\mathfrak{m}_x$, where $\mathfrak{m}_x$ is a maximal ideal of a point. From the exact sequence $0 \longrightarrow \mathfrak{m}_x \longrightarrow \mathcal{O} \longrightarrow k_x \longrightarrow 0$ we obtain that $c_1(k_x) = 1$ **(prove it)**. This proves the formula (*) for $F = k_x$, and hence for all coherent sheaves of rank 0.

Step 3: Let L be a line bundle. Any holomorphic section of L defines an exact sequence $0 \longrightarrow \mathcal{O} \longrightarrow L \longrightarrow F \longrightarrow 0$ where F is a torsion sheaf. For F and $\mathcal{O}$ the Riemann-Roch formula (*) is already proven, and **this implies that (*) holds for L.**

Riemann-Roch theorem for complex curves (2)

Step 4: The Riemann-Roch formula (*) is also true for any bundle L such that its dual L^* has non-zero holomorphic sections. Indeed, the section gives a sheaf monomorphism $\mathcal{O} \longrightarrow L^*$; tensoring with L, we obtain a monomorphism $L \longrightarrow \mathcal{O}$, giving an exact sequence $0 \longrightarrow L \longrightarrow \mathcal{O} \longrightarrow F \longrightarrow 0$ with $\operatorname{rk} F = 0$. **Since the Riemann-Roch formula (*) is already proven for two terms of this exact sequence, it is true for the third.**

Step 5: Now we can prove (*) for any line bundle L on M. Let L_1 be a very ample bundle on M. Since L_1 admits holomorphic sections, the Riemann-Roch formula (*) is true for L_1 and its powers. Replacing L_1 for a sufficiently big power and applying Kodaira vanishing, we obtain that $L \otimes L_1$ has sections, giving an embedding $\mathcal{O} \hookrightarrow L \otimes L_1$. Tensoring with L_1^*, we obtain a sheaf monomorphism which leads to an exact sequence $0 \longrightarrow L_1^{-1} \longrightarrow L \longrightarrow F \longrightarrow 0$, where F is a rank 1 sheaf. Again, **since the Riemann-Roch formula (*) is already proven for two terms of this exact sequence, and therefore it is also true for the third.** ■

Riemann-Roch-Hirzebruch for line bundles on complex surfaces

DEFINITION: A complex surface is a compact complex manifold of dimension 2.

REMARK: In the sequel, we will sometimes write L to denote $c_1(L)$ for a line bundle L. The fundamental class $[D]$ of a divisor D is sometimes denoted D. Note that $c_1(\mathcal{O}(D)) = [D]$ **(prove this)**; this implies that this set of conventions is consistent.

We prove the following weaker version of Riemann-Roch-Hirzebruch formula on surfaces.

THEOREM: (Riemann-Roch-Hirzebruch formula on surfaces)
Let L be a line bundle on a complex surface X, and $K_X := \Omega^2 X$ its canpnical bundle. **Then**

$$\chi(L) = \chi(\mathcal{O}_X) + \frac{(L - K_X, L)}{2}, \qquad (**)$$

where (A, B) denotes the intersection form applied to cohomology classes on X.

Riemann-Roch-Hirzebruch for line bundles on complex surfaces (2)

THEOREM: (Riemann-Roch-Hirzebruch formula on surfaces) Let L be a line bundle on a complex surface X, and $K_X := \Omega^2 X$ its canpnical bundle. **Then**

$$\chi(L) = \chi(\mathcal{O}_X) + \frac{(L - K_X, L)}{2}, \qquad (**)$$

where (A, B) denotes the intersection form applied to cohomology classes on X.

Proof. Step 1: Let $0 \longrightarrow L_1 \longrightarrow L_2 \longrightarrow L_2|_D \longrightarrow 0$ be an exact sequence, where L_i are line bundles and D a smooth curve of genus g. **The Riemann-Roch formula for curves gives** $\chi(L_1) = \chi(L_2) + (L_2, D) + (1 - g)$, indeed, $c_1(L_2|_D) = \deg_D L_2 = (L_2, D)$.

Step 2: Let ND denote the normal bundle of D. The adjunction formula gives $K_D = K_X|_D \otimes ND$. Since $g - 1 = \deg K_D/2$, we obtain $1 - g = -(K_X + D, D)/2$.

Riemann-Roch-Hirzebruch for line bundles on complex surfaces (3)

Step 3: Let $\chi'(L)$ be the RHS of (**), $\chi'(L) := \chi(\mathcal{O}_X) + (L - K_X, L)/2$. In assumptions of Step 1, we have $c_1(L_2) = c_1(L_1) + D$, giving

$$\chi'(L_2) - \chi'(L_1) = \frac{1}{2}\Big[(L_2 - K_X, L_2) - (L_2 - K_X - D, L_2 - D)\Big] = (L_2, D) - (K_X + D, D)/2.$$

Step 4: Comparing the statements of step 2 and step 3, we obtain $\chi'(L_2) - \chi'(L_1) = \chi(L_2) - \chi(L_1)$. Therefore, **(**) for L_2 is equivalent to (**) for L_1.**

Step 5: For any ample bundle L, the bundle $L \otimes A^{\otimes N}$ has smooth sections $N \gg 0$ by Bertini theorem, giving an exact sequence $0 \longrightarrow L \longrightarrow L \otimes A^{\otimes N+1} \longrightarrow L \otimes A^{\otimes N+1}|_D \longrightarrow 0$. **By Step 4, it remains to prove (**) for the line bundle $L \otimes A^{\otimes N+1}$, which can be chosen very ample.**

Step 6: For a very ample bundle L, we have an exact sequence

$$0 \longrightarrow \mathcal{O}_X \longrightarrow L \longrightarrow L|_D \longrightarrow 0,$$

where D is the zero set of a general section $\nu \in H^0(X, L)$. **Since D is smooth, and (*) is trivially true for $L = \mathcal{O}_X$, this proves (**) for L.** ■