# K3 surfaces

## lecture 6: Teichmüller spaces and the local Torelli theorem

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## Exponential exact sequence

The exponential exact sequence

$$0 \longrightarrow \mathbb{Z}_M \longrightarrow \mathcal{O}_M \longrightarrow \mathcal{O}_M^* \longrightarrow 0,$$

**gives a long exact sequence**

$$H^1(\mathcal{O}_M) \longrightarrow H^1(\mathcal{O}_M^*) \longrightarrow H^2(M,\mathbb{Z}) \xrightarrow{\alpha} H^2(\mathcal{O}_M).$$

The group $H^2(\mathcal{O}_M)$ is identified with $H^{0,2}(M)$, hence the kernel of $\alpha$ is $H^2(M,\mathbb{Z}) \cap H^{1,1}(M)$.

This gives

**PROPOSITION: The set of first Chern classes of holomorphic line bundles on a Kähler manifold is $H^{1,1}(M) \cap H^2(M,\mathbb{Z})$.**

## K3 surfaces are holomorphically symplectic

**DEFINITION: A complex surface** is a compact, complex manifold of complex dimension 2.

**DEFINITION: A K3 surface** is a Kähler complex surface $M$ with $b_1 = 0$ and $c_1(M, \mathbb{Z}) = 0$.

**REMARK: All surfaces with $b_1$ even are Kähler** (Kodaira, Buchdahl-Lamari).

**REMARK:** Since $b_1(M) = 0$, for a K3 we have $H^1(\mathcal{O}_M) = 0$ (follows from Hodge theory). **The canonical bundle of a K3 surface is trivial.** This follows from the exponential exact sequence $0 = H^1(\mathcal{O}_M) \longrightarrow \operatorname{Pic}^1(M) \xrightarrow{c_1} H^2(M, \mathbb{Z})$

**COROLLARY: A K3 surface is holomorphically symplectic.**

## Hodge diamond of a K3 surface

**REMARK:** By Serre's duality (and from the triviality of $K_M$) we have $H^2(\mathcal{O}_M) = H^0(K_M)^* = \mathbb{C}$. Since $H^1(\mathcal{O}_M) = 0$, this gives $\chi(\mathcal{O}_M) = 2$. Riemann-Roch-Hirzebruch formula gives $2 = \chi(\mathcal{O}_M) = \frac{c_2(M)}{12}$, hence $c_2(M) = 0$. Since $c_2(M)$ is equal to the topological Euler characteristic of $M$, **this implies $b_2(M) = 22$.**

The Hodge diamond for a K3 surface:

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

## Geometric structures

**DEFINITION: "Geometric structure"** on a manifold $M$ is a reduction of its structure group $GL(n,\mathbb{R})$ to a subgroup $G \subset GL(n,\mathbb{R})$. However, it is easier to define it by a collection of tensors $\Psi_1,...,\Psi_n$ such that the stabilizer $\mathrm{St}_{\langle\Psi_1,...,\Psi_n\rangle} \subset GL(T_xM)$ of $\Psi_1,...,\Psi_n$ at each point $x \in M$ **is conjugate to the same group $G \subset GL(n,\mathbb{R})$.** Usually, in addition to this algebraic condition, people ask for some differential conditions to hold, such as the integrability for almost complex structures.

**DEFINITION:** Let $M$ be a smooth manifold. An **almost complex structure** is an operator $I: TM \longrightarrow TM$ which satisfies $I^2 = -\mathrm{Id}_{TM}$.

The eigenvalues of this operator are $\pm\sqrt{-1}$. The corresponding eigenvalue decomposition is denoted $TM \otimes \mathbb{C} = T^{0,1}M \oplus T^{1,0}(M)$.

**DEFINITION:** An almost complex structure is **integrable** if $[T^{0,1}M, T^{0,1}M] \subset T^{0,1}M$.

**DEFINITION: Symplectic form** on a manifold is a non-degenerate differential 2-form $\omega$ satisfying $d\omega = 0$.

Today I would define **the Techmüller space** of geometric structures and describe it for some examples.

## Fréchet spaces

**DEFINITION: A seminorm** on a vector space $V$ is a function $\nu: V \longrightarrow \mathbb{R}^{\geqslant 0}$ satisfying

1. $\nu(\lambda x) = |\lambda|\nu(x)$ for each $\lambda \in \mathbb{R}$ and all $x \in V$

2. $\nu(x+y) \leqslant \nu(x) + \nu(y)$.

**DEFINITION:** We say that **topology on a vector space $V$ is defined by a family of seminorms $\{\nu_\alpha\}$** if the base of this topology is given by the finite intersections of the sets

$$B_{\nu_\alpha,\varepsilon}(x) := \{y \in V \ \ | \ \ \nu_\alpha(x-y) < \varepsilon\}$$

("open balls with respect to the seminorm"). It is **complete** if each sequence $x_i \in V$ which is Cauchy with respect to each of the seminorms converges.

**CLAIM:** A topology on $V$ defined by a family of seminorms $\{\nu_\alpha\}$ **is Hausdorff if and only if for each $v \in V$ there exists a seminorm $\nu \in \{\nu_\alpha\}$ such that $\nu(v) \neq 0$.** ■

## Fréchet spaces and translation-invariant metrics

**DEFINITION:** A **Fréchet space** is a Hausdorff second countable topological vector space $V$ with the topology which can be defined by a countable family of seminorms, complete with respect to this family of seminorms.

**DEFINITION:** Equivalent definition: let $V$ be a vector space equipped with a collection of norms (or seminorms) $|\cdot|_i$, $i = 0, 1, 2, ...$ and a topology which is given by the metric $d(x, y) = \sum_{i=0}^{\infty} 2^{-i} \min(|x - y|_i, 1)$, assumed to be non-degenerate. The space $V$ is called **a Fréchet space** if this metric is complete.

**REMARK:** Completeness **is equivalent to convergence of any sequence $\{a_i\}$ which is fundamental with respect to all the (semi-)norms $|\cdot|_i$.**

**REMARK: A sequence converges in the Fréchet topology given by $d$ $\Leftrightarrow$ it converges in any of the (semi-)norms $|\cdot|_i$.**

**EXERCISE:** Let $V$ be a vector space, equipped with a translation-invariant metric $d$. Assume that the open balls are convex, and $V$ is complete and second countable with respect to $d$. **Prove that $V$ is Fréchet, and all Fréchet spaces can be obtained this way.**

## $C^\infty$-topology

**DEFINITION:** Let $M$ be a Riemannian manifold, and $\nabla^i : C^\infty(M) \longrightarrow \Lambda^1(M)^{\otimes i}$ the iterated connection. **Topology $C^k$** on the space $C^\infty_c(M)$ of functions with compact support is defined by the norm

$$|\varphi|_{C^k} := \sup_M \sum_{i=0}^k |\nabla^i \varphi|.$$

**EXERCISE:** Prove that the space $C^\infty_c M$ of functions with compact support **is a Fréchet space with respect to $C^\infty$-topology.**

**REMARK:** This topology **is independent from the choice of the connection.** This is an exercise.

**REMARK: A tensor** on a manifold is a section of the tensor bundle $TM^{\otimes i} \otimes T^*M^{\otimes j}$. The same way one defines **the $C^\infty$-topology on the space of tensors with compact support on $M$.**

**EXERCISE:** Prove that **the space of tensors with compact support is a Fréchet space,** with the $C^\infty$-topology defined as above.

## $C^0$-topology on the group of diffeomorphisms

**DEFINITION:** Let $M$ be a compact Riemannian manifold. **The $C^0$-topology** on the space of diffeomorphisms is defined by the metric $d(\tau_1, \tau_2) := \sup_{x\in M} d(\tau_1(x), \tau_2(x))$.

**EXERCISE:** Prove that **this topology is independent from the choice of Riemannian structure.**

**EXERCISE:** Prove that **the group of homeomorphisms is complete with respect to $d$.**

**REMARK:** This topology is not enough for many purposes, for example, **the map $\tau \longrightarrow D_x\tau$ is not continuous in $C^0$-topology,** because it depends on the derivative of the diffeomorphism.

## $C^\infty$-topology on the group of diffeomorphisms

We define **$C^\infty$-topology** on diffeomorphisms; it is **strictly stronger** (has more open sets) than the $C^0$-topology. We define it in such a way that the group structure on $\operatorname{Diff}(M)$ is compatible with the $C^\infty$-topology. Then **it would suffice to define topology on a sufficiently small $C^0$-neighbourhood of $\operatorname{Id} \in \operatorname{Diff}(M)$.**

**DEFINITION:** Choose two atlases $\{U_i\}$ and $\{V_i\}$ on $M$, with **the closure of each $U_i$ compact in $V_i$.** Then **there exists a $C^0$-neighbourhood $\mathcal{U}$ of $\operatorname{Id} \in \operatorname{Diff}(M)$ such that for all $\tau \in \mathcal{U}$ we have $\tau(U_i) \subset V_i$.** We define the $C^\infty$-topology on $\mathcal{U}$ and expand it to $\operatorname{Diff}(M)$ using the group structure. For each $\tau \in \mathcal{U}$, we can interpret $\tau$ as a map from $U_i$ to $V_i$, that is, as a collection of smooth functions. **The $C^\infty$-topology on $U$ is defined by uniform convergence of these functions with all their derivatives,** that is, by $C^\infty$-topology on $\prod_i C^\infty(U_i, V_i)$.

**THEOREM:** **The $C^\infty$-topology on diffeomorphisms is independent from the choices we made. The diffeomorphism group with respect to this topology is a Fréchet-Lie group.**

**Proof:** Left as an exercise.

## Teichmüller space of geometric structures

Let $\mathcal{C}$ be the set of all geometric structures of a given type, say, complex, or symplectic. We put $C^\infty$-topology (the topology of uniform convergence with all derivatives) on $\mathcal{C}$. Let $\operatorname{Diff}_0(M)$ be the connected component of its diffeomorphism group $\operatorname{Diff}(M)$ (**the group of isotopies**).

**DEFINITION:** The quotient $\mathcal{C}/\operatorname{Diff}_0$ is called **Teichmüller space** of geometric strictures of this type.

**DEFINITION:** The group $\Gamma := \operatorname{Diff}(M)/\operatorname{Diff}_0(M)$ is called **the mapping class group** of $M$. It acts on Teich by homeomorphisms.

**DEFINITION:** The orbit space $\mathcal{C}/\operatorname{Diff} = \operatorname{Teich}/\Gamma$ is called **the moduli space** of geometric structure of this type.

Today I will describe Teich in some interesting cases.

## Teichmüller space for symplectic structures

**DEFINITION:** Let $\Gamma(\Lambda^2 M)$ be the space of all 2-forms on a manifold $M$, and $\operatorname{Symp} \subset \Gamma(\Lambda^2 M)$ the space of all symplectic 2-forms. We equip $\Gamma(\Lambda^2 M)$ with $C^\infty$-topology of uniform convergence on compacts with all derivatives. Then $\Gamma(\Lambda^2 M)$ is a Frechet vector space, and Symp a Frechet manifold.

**DEFINITION:** Consider the group of diffeomorphisms, denoted Diff or $\operatorname{Diff}(M)$, as a Frechet Lie group, and denote its connected component ("group of isotopies") by $\operatorname{Diff}_0$. The quotient group $\Gamma := \operatorname{Diff} / \operatorname{Diff}_0$ is called **the mapping class group** of $M$.

**DEFINITION:** **Teichmüller space of symplectic structures on $M$** is defined as a quotient $\operatorname{Teich}_s := \operatorname{Symp} / \operatorname{Diff}_0$. The quotient $\operatorname{Teich}_s / \Gamma = \operatorname{Symp} / \operatorname{Diff}$, is called **the moduli space of symplectic structures.**

**REMARK:** In many cases $\Gamma$ acts on $\operatorname{Teich}_s$ with dense orbits, hence **the moduli space is not always well defined.**

**DEFINITION:** Two symplectic structures are called **isotopic** if they lie in the same orbit of $\operatorname{Diff}_0$.

## Moser's theorem

**DEFINITION:** Define **the period map** $\operatorname{Per}: \operatorname{Teich}_s \longrightarrow H^2(M, \mathbb{R})$ mapping a symplectic structure to its cohomology class.

**THEOREM: (Moser, 1965)**
The **Teichmüler space** $\operatorname{Teich}_s$ **is a manifold** (possibly, non-Hausdorff), and the **period map** $\operatorname{Per}: \operatorname{Teich}_s \longrightarrow H^2(M, \mathbb{R})$ **is locally a diffeomorphism.**

The proof is based on another theorem of Moser.

**Theorem 1: (Moser)**
Let $\omega_t$, $t \in S$ be a smooth family of symplectic structures, parametrized by a connected manifold $S$. Assume that the cohomology class $[\omega_t] \in H^2(M)$ is constant in $t$. **Then all $\omega_t$ are isotopic.**

**Proof of Moser theorem:** The period map $P: \operatorname{Symp} \longrightarrow H^2(M, \mathbb{R})$ is a smooth submersion. By Theorem 1, the conneced components of the fibers of $P$ are orbits of $\operatorname{Diff}_0(M)$. Therefore, Per is locally a diffeomorphism. ■

## Symplectic structures on a compact torus

**DEFINITION:** A symplectic structure $\omega$ on a torus is called **standard** if there exists a flat torsion-free connection preserving $\omega$.

**REMARK:** Moser's theorem immediately implies that **the set** $\operatorname{Teich}_{st}$ **of standard symplectic structures is open in the Teichmüller space.** Indeed, the period map from $\operatorname{Teich}_{st}$ to $H^2(M)$ is also locally a diffeomorphism.

**REMARK:** **It is not known if any non-standard symplectic structures exist** (even in dimension =4).

**THEOREM:** Let $\Lambda^2_{nd}(H_1(T)) \subset H^2(T)$ be the space of symplectic forms on $H_1(T)$, where $T$ is an even-dimensional torus. Consider the period map $\operatorname{Per}: \operatorname{Teich}_{st} \longrightarrow \Lambda^2_{nd}(H_1(T)) \subset H^2(T)$, where $\operatorname{Teich}_{st}$ is the Teichmüller space of standard symplectic structures on $T$. **Then** $\operatorname{Per}$ **is a diffeomorphism on each connected component of** $\operatorname{Teich}_{st}$**.**

**Proof:** Left as an exercise.

## The kernel of a differential form

**DEFINITION:** Let $\Omega$ be a differential form on $M$. The **kernel**, or **the null-space** $\ker(\Omega) \subset TM$ of $\Omega$ is the space of all vector fields $X \in TM$ such that the contraction $i_X(\Omega)$ vanishes.

**Proposition 1:** Let $\Omega$ be a closed form on a manifold, and $B \subset TM$ its null-space. **Then $[B, B] \subset B$.**

**Proof. Step 1:** Let $X, X_1 \in \ker(\Omega)$, and $X_2, ..., X_p$ any vector fields. Cartan's formula implies that $\operatorname{Lie}_X(\Omega) = d(i_X(\Omega)) + i_X(d\Omega) = 0$, hence $\operatorname{Lie}_X(\Omega) = 0$.

**Step 2:** $\operatorname{Lie}_X(\Omega)(X_1, ..., X_p) = \operatorname{Lie}_X(\Omega(X_1, ..., X_p)) - \sum_{i=1}^p \Omega(X_1, ..., [X, X_i], ...X_p)$. All terms of this sum, except $\Omega([X, X_1], X_2, ..., X_p)$, vanish, because $X_1 \in \ker(\Omega)$. Since $\operatorname{Lie}_X(\Omega) = 0$, we have $\Omega([X, X_1], X_2, ..., X_p) = 0$ for all $X_2, ..., X_p$. Therefore, $[X, X_1] \in \ker(\Omega)$. ■

## Holomorphic symplectic form

**DEFINITION: Holomorphic symplectic form** on an almost complex manifold $(M,I)$ is a non-degenerate closed differential 2-form $\Omega \in \Lambda^2(M,\mathbb{C})$ satisfying $d\Omega = 0$ and $\Omega(Ix,y) = \sqrt{-1}\,\Omega(x,y)$.

**REMARK:** Consider the Hodge decomposition $T_{\mathbb{C}}M = T^{1,0}M \oplus T^{0,1}M$ (decomposition according to eigenvalues of $I$). Since $\Omega(IX,Y) = \sqrt{-1}\,\Omega(X,Y)$ and $I(Z) = -\sqrt{-1}\,Z$ for any $Z \in T^{0,1}(M)$, we have $\ker(\Omega) \supset T^{0,1}(M)$. Since $\ker\Omega \cap T_{\mathbb{R}}M = 0$, real dimension of its kernel is at most $\dim_{\mathbb{R}} M$, giving $\dim_{\mathbb{R}} \ker\Omega = \dim M$. **Therefore,** $\ker(\Omega) = T^{0,1}M$.

**COROLLARY:** Let $(M,I)$ be an almost complex manifold admitting a holomorphic symplectic form. **Then $I$ is integrable** (Proposition 1.)

**COROLLARY:** Let $\Omega$ be a holomorphically symplectic form on a complex manifold $(M,I)$. **Then $I$ is determined by $\Omega$ uniquely.**

## C-symplectic structures

**DEFINITION:** Let $M$ be a smooth $4n$-dimensional manifold. A closed complex-valued form $\Omega$ on $M$ is called **C-symplectic** if $\Omega^{n+1}=0$ and $\Omega^n\wedge\overline{\Omega}^n$ is a non-degenerate volume form.

**THEOREM:** Let $\Omega\in\Lambda^2(M,\mathbb{C})$ be a C-symplectic form, and $T^{0,1}_\Omega(M)$ be equal to $\ker\Omega$, where

$$\ker\Omega := \{v\in TM\otimes\mathbb{C}\ \ |\ \ \Omega\lrcorner v=0\}.$$

Then $T^{0,1}_\Omega(M)\oplus\overline{T^{0,1}_\Omega(M)}=TM\otimes_{\mathbb{R}}\mathbb{C}$, hence **the sub-bundle $T^{0,1}_\Omega(M)$ defines an almost complex structure $I_\Omega$ on $M$.** If, in addition, **$\Omega$ is closed, $I_\Omega$ is integrable, and $\Omega$ is holomorphically symplectic on $(M,I_\Omega)$.**

**Proof:** Rank of $\Omega$ is $2n$ because $\Omega^{n+1}=0$ and $\Omega^n\wedge\overline{\Omega}^n$ is non-degenerate. Therefore, $\operatorname{rk}T^{0,1}_\Omega(M)=2n$. For any $v\in T^{0,1}_\Omega(M)\cap\overline{T^{0,1}_\Omega(M)}$, the real part $\operatorname{Re}v$ of $v$ belongs to $\ker\Omega$. This would imply that $\operatorname{Re}v\in\operatorname{Im}(\Omega^n\wedge\overline{\Omega}^n)$, which is impossible, because $\Omega^n\wedge\overline{\Omega}^n$ is non-degenerate. Then $T^{0,1}_\Omega(M)\oplus\overline{T^{0,1}_\Omega(M)}=TM\otimes_{\mathbb{R}}\mathbb{C}$, defining an almost complex structure $I_\Omega$. Its integrabiliuty immediately follows from Proposition 1. ■

## C-symplectic structures on a K3 surface

## C-symplectic structures on surfaces

**CLAIM:** In real dimension 4, a C-symplectic structure **is determined by a pair $\omega_1 = \operatorname{Re}\Omega, \omega_2 = \operatorname{Im}\Omega$ of symplectic forms which satisfy $\omega_1^2 = \omega_2^2$ and $\omega_1 \wedge \omega_2 = 0$.**

**Proof:** Let $\Omega$ be a C-symplectic form, $\omega_1 = \operatorname{Re}\Omega$ and $\omega_2 = \operatorname{Im}\Omega$. Then $\Omega \wedge \Omega = \omega_1^2 + \omega_2^2 + 2\sqrt{-1}\,\omega_1 \wedge \omega_2 = 0$, hence $\omega_1^2 = \omega_2^2$ and $\omega_1 \wedge \omega_2 = 0$. The form $\Omega \wedge \overline{\Omega} = \omega_1^2 + \omega_2^2$ is non-degenerate, hence $\omega_1^2 = \omega_2^2$ is non-degenerate.

Conversely, if $\omega_1^2 = \omega_2^2$ and $\omega_1 \wedge \omega_2 = 0$, we have $\Omega \wedge \Omega = 0$, and $\Omega \wedge \overline{\Omega} = \omega_1^2 + \omega_2^2$ is non-degenerate if $\omega_i$ is non-degenerate. ■

**REMARK:** For K3 surface, the local Torelli theorem would imply that the period map Per : CTeich $\longrightarrow H^2(M, \mathbb{C})$ takes $\omega_1, \omega_2$ to their cohomology classes which satisfy $[\omega_1]^2 = [\omega_2]^2$ and $[\omega_1] \wedge [\omega_2] = 0$.

## Period map for holomorphically symplectic manifolds

**DEFINITION:** Let $(M, I, \Omega)$ be a holomorphically symplectic manifold, and CSymp the space of all C-symplectic forms. The quotient $\text{CTeich} := \frac{\text{CSymp}}{\text{Diff}_0}$ is called **the holomorphically symplectic Teichmüller space**, and the map $\text{CTeich} \longrightarrow H^2(M, \mathbb{C})$ taking $(M, I, \Omega)$ to the cohomology class $[\Omega] \in H^2(M, \mathbb{C})$ **the holomorphically symplectic period map**.

We want to prove that **the period map is locally an embedding.** This is immediately implied by the following version of Moser's lemma.

**THEOREM:** Let $(M, I_t, \Omega_t)$, $t \in [0, 1]$ be a family of C-symplectic forms on a compact manifold. Assume that the cohomology class $[\Omega_t] \in H^2(M, \mathbb{C})$ is constant, and $H^{0,1}(M, I_t) = 0$, where $H^{0,1}(M, I_t) = H^1(M, \mathcal{O}_{(M,I_t)})$ is cohomology of the sheaf of holomorphic functions. Then **there exists a smooth family of diffeomorphisms $V_t \in \text{Diff}_0(M)$, such that $V_t^* \Omega_0 = \Omega_t$.**

**Proof:** Later in this course.

## Local Torelli theorem

**REMARK:** In real dimension 4, C-symplectic form **is a pair $\omega_1, \omega_2$ of symplectic forms which satisfy $\omega_1^2 = \omega_2^2$ and $\omega_1 \wedge \omega_2 = 0$.**

**THEOREM:** Let $(M, I, \Omega)$ be a complex holomorphically symplectic surface with $H^{0,1}(M) = 0$, that is, a K3 surface. Consider the period map $\operatorname{Per}: \operatorname{CTeich} \longrightarrow H^2(M, \mathbb{C})$ taking $(M, I, \Omega)$ to the cohomology class $[\Omega] \in H^2(M, \mathbb{C})$. **Then** Per **is a local diffeomorpism** of CTeich to the **period space** $Q := \{v \in H^2(M, \mathbb{C}) \quad | \quad \int_M v \wedge v = 0, \int_M v \wedge \overline{v} > 0\}$.

**Proof:** Later in this course.

**A caution:** CTeich is smooth, but non-Hausdorff.

## The period space of complex structures

**DEFINITION:** Since $H^{2,0}(M) = \mathbb{C}$, the space CTeich is $\mathbb{C}^*$-fibered over the space Teich of complex structures on K3. The corresponding period space is denoted $\mathbb{P}\mathrm{er} := \{v \in \mathbb{P}H^2(M, \mathbb{C}) \quad | \quad \int_M v \wedge v = 0, \int_M v \wedge \overline{v} > 0\}$.

The following theorem follows from local Torelli for C-symplectic structures.

**PROPOSITION: (local Torelli theorem for complex structures)**
Let Teich be the space of complex structures on a K3 surface, and Per : Teich $\longrightarrow \mathbb{P}\mathrm{er}$ the map taking $(M, I)$ to the line $H^{2,0}(M) \subset H^2(M, \mathbb{C})$. **Then Per is a local diffeomorphism.**

**Proof:** The group $\mathbb{C}^*$ acts on CTeich and on $Q$, which are locally diffeomorphic, hence Teich $=$ CTeich $/\mathbb{C}^*$ is locally diffeomorphic to $\mathbb{P}\mathrm{er} = Q/\mathbb{C}^*$.
■

## The period space of complex structures is a Grassmannian

**CLAIM:** $\mathbb{P}\mathrm{er} = SO(b_2 - 3, 3)/SO(2) \times SO(b_2 - 3, 1)$.

**Proof. First version:** Indeed, the group $SO(H^2(M, \mathbb{R}), q) = SO(b_2 - 3, 3)$ acts transitively on $\mathbb{P}\mathrm{er}$, and $SO(2) \times SO(b_2 - 3, 1)$ is a stabilizer of a point.

**Proof. Second version:** Take a non-zero vector $v$ in a line $l \in \mathbb{P}\mathrm{er}$. Since $(v, v) = 0$ and $(v, \overline{v}) > 0$, the vectors $v$ and $\overline{v}$ are not proportional, hence they generate a 2-dimensional plane $P \subset H^2(M, \mathbb{R})$ which is positive, because $(v, \overline{v}) > 0$, hence belongs to the positive, oriented Grassmannian

$$\mathrm{Gr}_{++}(H^2(M, \mathbb{R})) = SO(b_2 - 3, 3)/SO(2) \times SO(b_2 - 3, 1).$$

Conversely, for any $P \subset \mathrm{Gr}_{++}(H^2(M, \mathbb{R}))$, its complexification $P \otimes \mathbb{C}$ contains two lines $l_1, l_2$ which belong to the quadric $q(v, v) = 0$. These two lines are distinguished by their orientation. This implies that **the correspondence $\mathbb{P}\mathrm{er} \to SO(b_2 - 3, 3)/SO(2) \times SO(b_2 - 3, 1)$ taking $\Omega \in Q$ to $\langle \mathrm{Re}\,\omega, \mathrm{Im}\,\Omega \rangle \in \mathrm{Gr}_{++}(H^2(M, \mathbb{R}))$ is bijective.** ■