# K3 surfaces

## lecture 7: Intersection form of a K3 surface

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## K3 surfaces are holomorphically symplectic (reminder)

**DEFINITION: A complex surface** is a compact, complex manifold of complex dimension 2.

**DEFINITION: A K3 surface** is a Kähler complex surface $M$ with $b_1 = 0$ and $c_1(M, \mathbb{Z}) = 0$.

**REMARK: All surfaces with $b_1$ even are Kähler** (Kodaira, Buchdahl-Lamari).

**REMARK:** Since $b_1(M) = 0$, for a K3 we have $H^1(\mathcal{O}_M) = 0$ (follows from Hodge theory). **The canonical bundle of a K3 surface is trivial.** This follows from the exponential exact sequence $0 = H^1(\mathcal{O}_M) \longrightarrow \operatorname{Pic}^1(M) \xrightarrow{c_1} H^2(M, \mathbb{Z})$

**COROLLARY: A K3 surface is holomorphically symplectic.**

## Period map for holomorphically symplectic manifolds (reminder)

**DEFINITION:** Let $(M, I, \Omega)$ be a holomorphically symplectic manifold, and CSymp the space of all C-symplectic forms. The quotient $\mathrm{CTeich} := \frac{\mathrm{CSymp}}{\mathrm{Diff}_0}$ is called **the holomorphically symplectic Teichmüller space**, and the map $\mathrm{CTeich} \longrightarrow H^2(M, \mathbb{C})$ taking $(M, I, \Omega)$ to the cohomology class $[\Omega] \in H^2(M, \mathbb{C})$ **the holomorphically symplectic period map**.

We want to prove that **the period map is locally an embedding.** This is immediately implied by the following version of Moser's lemma.

**THEOREM:** Let $(M, I_t, \Omega_t)$, $t \in [0, 1]$ be a family of C-symplectic forms on a compact manifold. Assume that the cohomology class $[\Omega_t] \in H^2(M, \mathbb{C})$ is constant, and $H^{0,1}(M, I_t) = 0$, where $H^{0,1}(M, I_t) = H^1(M, \mathcal{O}_{(M,I_t)})$ is cohomology of the sheaf of holomorphic functions. Then **there exists a smooth family of diffeomorphisms $V_t \in \mathrm{Diff}_0(M)$, such that $V_t^* \Omega_0 = \Omega_t$.**

**Proof:** Later in this course.

## Intersection form on $\operatorname{Re}\Lambda^{2,0}(V)$

**Lemma A:** Let $(V, I, g)$ be a 4-dimensional space equipped with a complex structure operator $I \in \operatorname{End}(V)$, $I^2 = -\operatorname{Id}$, and $W := \operatorname{Re}(\Lambda^{2,0}(V,I)) \subset \Lambda^4(V)$ **Then for any non-zero $\alpha \in W$, one has $\frac{\alpha\wedge\alpha}{\operatorname{Vol}} > 0$.**

**Proof:** The space $\Lambda^{2,0}(V,I) \subset \Lambda^2(V \otimes_{\mathbb{R}} \mathbb{C})$ is 1-dimensional over $\mathbb{C}$. Let $\Omega = \omega_1 + \sqrt{-1}\,\omega_2$ be its generator, with $\omega_1 = \operatorname{Re}\Omega$, $\omega_2 = \operatorname{Im}\Omega$. Then

$$0 = \Omega \wedge \Omega = \omega_1^2 - \omega_2^2 + 2\sqrt{-1}\,\omega_1 \wedge \omega_2$$

implies $\omega_1^2 = \omega_2^2$. By definition of the orientation in $V$, one has $\frac{\Omega\wedge\overline{\Omega}}{\operatorname{Vol}} > 0$, but $\Omega \wedge \overline{\Omega} = \omega_1^2 + \omega_2^2 = 2\omega_2^2$, hence $\omega_1^2 > 0$. ■

## Intersection form on $\operatorname{Re}\Lambda^{1,1}_{prim}(V)$

**Lemma B:** Let $(V, I, g)$ be a 4-dimensional space equipped with a complex structure operator $I \in \operatorname{End}(V)$, $I^2 = -\operatorname{Id}$, $\omega \in \Lambda^{1,1}(V)$ a Hermitian form, and $\Lambda^{1,1}_{prim}(V) \subset \Lambda^{1,1}(V, I) \subset \Lambda^4(V)$ be the space of (1,1)-forms $\alpha$ such that $\alpha \wedge \omega = 0$. **Then for any non-zero $\alpha \in W$, one has $\frac{\alpha\wedge\alpha}{\mathsf{Vol}} < 0$.**

**Proof. Step 1:** Consider the Hodge star operator $* : \Lambda^2(V) \longrightarrow \Lambda^2(V)$. Clearly, $*^2 = \operatorname{Id}$, hence all eigenvalues of $*$ are $\pm 1$. If we invert the orientation, $*$ becomes $-*$; this implies that $*$ is conjugated to $-*$, hence the multiplicity of 1 and $-1$ is equal 3. **Denote the corresponding eigenspaces as $\Lambda^2 V = \Lambda^+ V \oplus \Lambda^- V$.** This decomposition is clearly orthogonal with respect to the pairing $\alpha, \beta \longrightarrow \frac{\alpha\wedge\beta}{\mathsf{Vol}}$.

**Step 2:** Let us identify $V$ with quaternionic algebra $\mathbb{H}$ Then the three symplectic structures $\operatorname{Re}\Omega, \operatorname{Im}\Omega, \omega$ can be understood as Hermitian forms for $I, J, K$, which implies that $*\omega = \omega$, and the same is true for $\operatorname{Re}\Omega, \operatorname{Im}\Omega$. **Therefore, $\langle \operatorname{Re}\Omega, \operatorname{Im}\Omega, \omega\rangle = \Lambda^+ V$.**

**Step 3:** The space $\Lambda^{1,1}_{prim}(V)$ is 3-dimensional and orthogonal to the 3-dimensional space $\langle \operatorname{Re}\Omega, \operatorname{Im}\Omega, \omega\rangle$. The space $\langle \operatorname{Re}\Omega, \operatorname{Im}\Omega, \omega\rangle$ is equal to $\Lambda^+ V$, as follows from Step 2. **Then $\Lambda^{1,1}_{prim}(V) = \langle \operatorname{Re}\Omega, \operatorname{Im}\Omega, \omega\rangle^\perp = \Lambda^- V$.** ■

## Hodge index formula

**THEOREM:** Let $(M, I, \omega)$ be a complex Kähler surface, and $H^{1,1}_{prim}(M)$ the kernel of the natural map $\alpha \to \int_M \alpha \wedge \omega$. Then **the intersection form is positive definite on the space $\operatorname{Re} H^{2,0}(M) \oplus \mathbb{R}\omega$ and negative definite in $H^{1,1}_{prim}(M)$.**

**Proof. Step 1:** It is clear that the intersection form is positive definite on the space $\langle\omega\rangle$, which is orthogonal to $\operatorname{Re} H^{2,0}(M)$. On $\operatorname{Re} \Lambda^{2,0}(M)$, the form $\alpha \to \alpha \wedge \alpha$ , taking value in $C^\infty M$, is positive by the previous lemma, **hence the intersection form is positive definite on $(2,0)+(0,2)$ real cohomology classes.** It remains only to prove it is negative definite on $H^{1,1}_{prim}(M)$.

**Step 2:** The operator $\alpha \to \alpha \wedge \omega$ takes harmonic form to harmonic, as follows from Hodge theory. Therefore, for any $[\alpha] \in H^{1,1}_{prim}(M)$, its harmonic representative $\alpha$ satisfies $\alpha \wedge \omega = 0$. Now, **Lemma B implies that the intersection form is negative definite on $H^{1,1}_{prim}(M)$.** ∎

## Local Torelli theorem

**REMARK:** In real dimension 4, C-symplectic form **is a pair $\omega_1, \omega_2$ of symplectic forms which satisfy $\omega_1^2 = \omega_2^2$ and $\omega_1 \wedge \omega_2 = 0$.**

**THEOREM:** Let $(M, I, \Omega)$ be a complex holomorphically symplectic surface with $H^{0,1}(M) = 0$, that is, a K3 surface. Consider the period map $\operatorname{Per}: \operatorname{CTeich} \longrightarrow H^2(M, \mathbb{C})$ taking $(M, I, \Omega)$ to the cohomology class $[\Omega] \in H^2(M, \mathbb{C})$. **Then** Per **is a local diffeomorpism** of CTeich to the **period space** $Q := \{v \in H^2(M, \mathbb{C}) \quad | \quad \int_M v \wedge v = 0, \int_M v \wedge \overline{v} > 0\}$.

**Proof:** Later in this course.

**A caution:** CTeich is smooth, but non-Hausdorff.

## The period space of complex structures

**DEFINITION:** Since $H^{2,0}(M)=\mathbb{C}$, the space CTeich is $\mathbb{C}^*$-fibered over the space Teich of complex structures on K3. The corresponding period space is denoted $\mathbb{P}\mathrm{er}:=\{v\in \mathbb{P}H^2(M,\mathbb{C}) \quad | \quad \int_M v\wedge v=0, \int_M v\wedge \overline{v}>0\}$.

The following theorem follows from local Torelli for C-symplectic structures.

**PROPOSITION: (local Torelli theorem for complex structures)**
Let Teich be the space of complex structures on a K3 surface, and Per : Teich $\longrightarrow \mathbb{P}\mathrm{er}$ the map taking $(M,I)$ to the line $H^{2,0}(M)\subset H^2(M,\mathbb{C})$. **Then Per is a local diffeomorphism.**

**Proof:** The group $\mathbb{C}^*$ acts on CTeich and on $Q$, which are locally diffeomorphic, hence Teich = CTeich$/\mathbb{C}^*$ is locally diffeomorphic to $\mathbb{P}\mathrm{er}=Q/\mathbb{C}^*$. ■

## The period space of complex structures is a Grassmannian

**CLAIM:** $\mathbb{P}\mathrm{er} = SO(b_2 - 3, 3)/SO(2) \times SO(b_2 - 3, 1)$.

**Proof. First version:** Indeed, the group $SO(H^2(M, \mathbb{R}), q) = SO(b_2 - 3, 3)$ acts transitively on $\mathbb{P}\mathrm{er}$, and $SO(2) \times SO(b_2 - 3, 1)$ is a stabilizer of a point.

**Proof. Second version:** Take a non-zero vector $v$ in a line $l \in \mathbb{P}\mathrm{er}$. Since $(v, v) = 0$ and $(v, \overline{v}) > 0$, the vectors $v$ and $\overline{v}$ are not proportional, hence they generate a 2-dimensional plane $P \subset H^2(M, \mathbb{R})$ which is positive, because $(v, \overline{v}) > 0$, hence belongs to the positive, oriented Grassmannian

$$\mathrm{Gr}_{++}(H^2(M, \mathbb{R})) = SO(b_2 - 3, 3)/SO(2) \times SO(b_2 - 3, 1).$$

Conversely, for any $P \subset \mathrm{Gr}_{++}(H^2(M, \mathbb{R}))$, its complexification $P \otimes \mathbb{C}$ contains two lines $l_1, l_2$ which belong to the quadric $q(v, v) = 0$. These two lines are distinguished by their orientation. This implies that **the correspondence $\mathbb{P}\mathrm{er} \to SO(b_2 - 3, 3)/SO(2) \times SO(b_2 - 3, 1)$ taking $\Omega \in Q$ to $\langle \mathrm{Re}\, \omega, \mathrm{Im}\, \Omega \rangle \in \mathrm{Gr}_{++}(H^2(M, \mathbb{R}))$ is bijective.** ■

## The set of all classes of type (1,1)

**Corollary 1:** Let $U \subset \operatorname{Teich}$ be an open neighbourhood, and $V \subset H^2(M,\mathbb{R})$ the set of all cohomology classes which are of type (1,1) for some $I \in U$. **Then $V$ is open in $H^2(M,\mathbb{R})$.**

**Proof. Step 1:** Let $I \in \operatorname{Teich}$ and $P \in \operatorname{Gr}_{++}(H^2(M,\mathbb{R}))$ be the corresponding 2-space, $P = \operatorname{Re}(H^{2,0}(M,I))$. Then $H^{1,1}(M) = P^\perp$.

**Step 2:** Consider a 2-dimensional positive subspace $P \in \operatorname{Gr}_{++}(H^2(M,\mathbb{R}))$ associated with $I \in U$. Since Teich is locally diffeomorphic to $\operatorname{Gr}_{++}(H^2(M,\mathbb{R}))$, **it would suffice to show that for some neighbourhood $U_1 \ni P$ in $\operatorname{Gr}_{++}(H^2(M,\mathbb{R}))$, the union $\bigcup_{P_1 \in U_1} P_1^\perp$ is open in $H^2(M,\mathbb{R})$.**

**Step 3:** Consider a non-zero element $y \in H^2(M,\mathbb{R})$, which belongs to a sufficiently small neighbourhood $U_x$ of $x \in P^\perp$, and let $P_y$ be the projection of $P$ to $y^\perp$. Since $y$ is close to $x$, this projection is non-degenerate, and its image is a positive 2-plane. This defines a map $\Phi : U_x \longrightarrow \operatorname{Gr}_{++}$. Let $W := \Phi(U_x) \cap \operatorname{Per}(U)$. By construction, the set $S$ of all $y \in H^2(M,\mathbb{R})$ such that $y \perp P'$ for some $P' \in W$ contains $U_x$, hence it is open in $H^2(M,\mathbb{R})$. However, $S$ is the union of $H^{1,1}(I)$ for all $I \in \operatorname{Teich}$ such that $\operatorname{Per}(I) \in \Phi(U_x) \cap \operatorname{Per}(U)$, hence it belongs to $\bigcup_{I\in U} H^{1,1}(M,I)$. ■

## Intersection form for a K3 surface

**Lemma 1:** Let $\eta$ be an odd intersection form on $V_{\mathbb{Z}} = \mathbb{Z}^n$, and let $\pi : V_{\mathbb{Z}} \backslash 0 \longrightarrow \mathbb{P}(V \otimes_{\mathbb{Z}} \mathbb{R})$ be the standard projection. Consider the set $R$ of odd vectors in $V$. **Then $\pi(R)$ is dense in $\mathbb{P}(V \otimes_{\mathbb{Z}} \mathbb{R})$.**

**Proof. Step 1:** Let $s \in V_{\mathbb{Z}} \backslash 0$ be any vector. To prove that $\pi(R)$ is dense, it will suffice to fine an element of $\pi(R)$ in any neighbourhood of $\pi(s)$.

**Step 2:** Let $r_0 \in R$. Then all vectors in the sequence $r_n := r_0 + 2ns$ are odd, and $\lim_i \pi(r_i) = s$. ■

**THEOREM: The intersection form of a K3 surface is even.**

**Proof. Step 1:** Going ad absurdum, assume that the intersection form of a K3 is odd. Using Corollary 1 and Lemma 1, **we obtain a complex structure $I$ and an odd vector $r \in H^{1,1}(M, I)$.** Indeed, by Corollary 1, the union $\bigcup_{I \in \text{Teich}} H^{1,1}(M, I)$ of the set of all $(1,1)$-vectors is open in $H^2(M, R)$ and by Lemma 1 there are odd vectors in any $\mathbb{R}^*$-invariant open subset of $H^2(M, R)$.

**Step 2:** Let $L$ be a line bundle on $(M, I)$ such that $c_1(L) = r$ has odd self-intersection. By Riemann-Roch-Hirzebruch formula, $\chi(L) = 2 + \frac{1}{2} \int_M r \wedge r$, hence **the self-intersection of $c_1(L)$ is even.** ■

## Voting

**Order of the next lectures (please vote):**

1. **Local Torelli** (3-4 lectures), then Lefschetz hyperplane section, then density of quartics.
2. **Density of quartics,** then Lefschetz hyperplane section, then local Torelli.
3. **Lefschetz hyperplane section,** density of quartics, local Torelli.