K3 surfaces

lecture 8: Smooth quartics

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Smooth quartics

DEFINITION: A smooth quartic is a smooth hypersurface in $\mathbb{C}P^n$ defined by an irreducible homogeneous polynomial of degree 4.

REMARK: By Euler formula, the canonical bundle on $\mathbb{C}P^n$ is $\mathcal{O}(-n-1)$. Adjunction formula applied to a smooth hypersurface $Z \subset \mathbb{C}P^n$ of degree m gives $N^*Z \otimes_{\mathcal{O}_Z} K_Z = K_{\mathbb{C}P^n}|_Z$, where $NZ = \mathcal{O}(m)|_Z$ is the normal bundle. **This gives $K_Z = \mathcal{O}(m-n-1)$.**

COROLLARY: A smooth quartic in $\mathbb{C}P^3$ has trivial canonical bundle. ■

REMARK: In the sequel, **"smooth quartics" will always mean smooth quartic surfaces.**

DEFINITION: Veronese embedding is the projective embedding $\mathbb{C}P^n \longrightarrow \mathbb{P}(H^0(\mathcal{O}(k)^*))$, defined by the line system $H^0(\mathcal{O}(k))$. In other words, **the Veronese embedding takes**

$$(t_0 : t_1 : ... : t_n) \textbf{ to} (P_0(t_0, ..., t_n) : P_1(t_0, ..., t_n) : ... : ...),$$

where $\{P_i\}$ denotes a basis in homogeneous monomials of degree k.

CLAIM: A smooth quartic is an intersection of a hyperplane and the image of 4-th Veronese embedding of $\mathbb{C}P^3$.

Smooth quartics and Lefschetz hyperplane section theorem

THEOREM: (Lefschetz hyperplane section theorem) Let $Z \subset \mathbb{C}P^n$ be a smooth projective submanifold of dimension m, and $H \subset \mathbb{C}P^n$ a hyperplane section transversal to Z. Then **for any $i < m-1$, the map of homotopy groups $\pi_i(Z \cap H) \longrightarrow \pi_i(Z)$ is an isomorphism.**

Proof: Later in this course.

COROLLARY: A smooth quartic Z is a K3 surface.

Proof: Since Z is a hyperplane section of the Veronese manifold, which is isomorphic to $\mathbb{C}P^3$, **Lefschetz theorem gives $\pi_1(Z) = \pi_1(\mathbb{C}P^3) = 0$;** its canonical bundle $K_Z = \mathcal{O}(4-4)|_Z = \mathcal{O}_Z$ vanishes, as shown above. ■

REMARK: It is not very hard to show that **smooth quartics are diffeomorphic** (later today).

REMARK: Soon enough, we will prove that **smooth quartics are dense in the Teichmuller space of K3 surfaces,** hence **all K3 are diffeomorphic.**

Smooth submersions

DEFINITION: Let $\pi : M \longrightarrow M'$ be a smooth map of manifolds. This map is called **submersion** if at each point of M the differential $D\pi$ is surjective, and **immersion** if it is injective.

CLAIM: Let $\pi : M \longrightarrow M'$ be a submersion. Then each $m \in M$ has a neighbourhood $U \cong V \times W$, where V, W are smooth and $\pi|_U$ **is a projection of** $V \times W = U \subset M$ **to** $W \subset M'$ **along** V**.**

Proof: Follows from the inverse function theorem. ■

THEOREM: ("Ehresmann's fibration theorem")
Let $\pi : M \longrightarrow M'$ be a smooth submersion of compact manifolds. **Prove that** π **is a locally trivial fibration.**

Proof: Next slide.

DEFINITION: Vertical tangent space $T_\pi M \subset TM$ of a submersion $\pi : M \longrightarrow M'$ is the kernel of $D\pi$.

Ehresmann connections

DEFINITION: Let $\pi : M \longrightarrow Z$ be a smooth submersion, with $T_\pi M$ **the bundle of vertical tangent vectors** (vectors tangent to the fibers of π). An **Ehresmann connection** on π is a sub-bundle $T_{\mathsf{hor}} M \subset TM$ such that $TM = T_{\mathsf{hor}} M \oplus T_\pi M$. The **parallel transport** along the path $\gamma : [0, a] \longrightarrow Z$ associated with the Ehresmann connection is a diffeomorphism

$$V_t : \pi^{-1}(\gamma(0)) \longrightarrow \pi^{-1}(\gamma(t))$$

obtained by integrating a vector field $A_{\mathsf{hor}} = \frac{dV_t}{dt} \in T_{\mathsf{hor}} M$ which is a preimage of a vector field $\frac{d}{dt}$ on γ.

CLAIM: Let $\pi : M \longrightarrow Z$ be a smooth fibration with compact fibers. Then **the parallel transport, associated with the Ehresmann connection, always exists.**

Proof: Follows from existence and uniqueness of solutions of ODEs. ∎

Space of smooth quartics

Let $V = \mathbb{C}^{35}$ be the space of all homogeneous degree 4 polynomials in 4 variables. We interpret each non-zero $P \in V$ as a quartic equation in $W = \mathbb{C}^4$.

CLAIM: Let $Z \subset \mathbb{P}V \times \mathbb{C}P^3$ be the set of all pairs $\{(P \in \mathbb{P}V, w \in \mathbb{C}P^3) \mid P(w) = 0\}$. **Then Z is smooth and irreducible.**

Proof. Step 1: Clearly, Z is the space of all pairs $(x \in Q, l \ni x)$, where x is a point in Veronese quartic $Q = \mathbb{C}P^3 \subset \mathbb{C}P^{34}$, and l a hyperplane section passing through x. Let $\tilde{Z} \subset \mathbb{C}^4 \times \mathbb{C}^{35}$ be the corresponding set of vectors, so that $Z = \tilde{Z}/\mathbb{C}^* \times C^*$. **Clearly, it suffices to show that $\tilde{Z}$ is smooth.**

Step 2: Consider the evaluation map $\varphi: \ \tilde{Z} \longrightarrow \mathbb{C}$ taking a pair (P, w) to $P(w)$. The differential of this map is surjective everywhere, because for each non-zero w we can choose a 4-th order polynomial Q such that $Q(w) \neq 0$ and then $\frac{d}{dt}(P + tQ)(w) \neq 0$. **Therefore, $\tilde{Z}$ is smooth.**

Step 3: By Sard's lemma (or Bertini theorem, which is the same) Z projects to $\mathbb{C}P^{34}$ with the general fiber F which is a smooth quartic. By Lefschetz hyperplane section theorem, F is connected, **hence Z is irreducible** (it surjectively projects to $\mathbb{C}P^{34}$ with general fiber irreducible). ■

Smooth quartics are diffeomorphic

COROLLARY: **Smooth quartics are diffeomorphic**.

Proof. **Step 1:** The space of singular quartics is a proper algebraic subvariety in the space of all quadrics $\mathbb{P}V$, $V = \mathbb{C}^{35}$, hence **its complement U is connected.** Indeed, a complement to a divisor in a smooth manifold is always connected.

Step 2: Let $Z_{\text{sing}} \subset Z$ be the set of all pairs $(P, w) \in Z$ such that the corresponding quartic is singular. Then **all smooth quartics are fibers of a proper submersion from $Z \backslash Z_{\text{sing}}$ to U.** They are diffeomorphic by Ehresmann theorem. ■

REMARK: The same argument **shows that smooth hypersurfaces of degree d in $\mathbb{C}P^n$ are diffeomorphic.**