# K3 surfaces

## lecture 9: Nakai-Moishezon theorem

Misha Verbitsky

**IMPA, sala 236**

**September 30, 2024, 17:00**

## Ample bundles

**DEFINITION: Very ample bundle** on $M$ is a line bundle obtained as $\varphi^*(\mathcal{O}(1))$, where $\varphi: M \hookrightarrow \mathbb{C}P^n$ is a projective embedding. **Ample bundle** is a line bundle $L$ such that its tensor power $nL := L^{\otimes n}, n \geqslant 1$ is very ample.

**THEOREM: (Kodaira) A bundle $L$ is ample if and only if $c_1(L)$ is a Kähler class.**

**COROLLARY: Any line bundle of positive degree on a compact complex curve is ample.**

**REMARK:** To prove that all K3 are diffeomorphic, **we need to show that quartics are dense in the universal family of K3 over its Teichmüller space.** This means that **we need to identify the quartics among all K3 surfaces $M$ containing $x \in \operatorname{Pic}(M)$ such that $x \cap x = 4$;** in other words, we need to find **when line bundle $L$ with $c_1(L) \cap c_1(L) = 4$ is very ample.**

## Very ample bundles

**CLAIM:** Let $L$ be a line bundle on a compact complex manifold $X$. Then the following are equivalent:

(i) for any two distinct points $x, y \in X$ **there exists a section** $h \in H^0(X, L)$ **with zero divisor** $D \subset X$ **such that** $x \in D, y \notin D$**.**

(ii) The standard map $\varphi_L : X \longrightarrow \mathbb{P}(H^0(X, L)^*)$ **is holomorphic and injective.** ■

**REMARK:** Denote by $\mathfrak{m}_x$ the maximal ideal of $x$. Then the space of 1-jets of functions in $x$ is $\mathcal{O}_X/\mathfrak{m}_x^2$. The map $\varphi_L$ has non-degenerate differential if for each $x \in X$ there exist a section $h \in H^0(X, L)$ which vanishes in $x$ and has non-zero differential in $x$, in other words, **when the natural map** $H^0(X, L) \longrightarrow H^0(X, L \otimes \mathcal{O}_X/\mathfrak{m}_x^2)$ **is surjective.**

## Very ample bundles (2)

**COROLLARY:** Let $L$ be a bundle on a compact complex manifold $X$. Then the following are equivalent:

(i) $L$ **is very ample**

(ii) for any two distinct points $x, y \in X$, **the natural maps**

$$H^0(X, L) \longrightarrow H^0\left(X, L \otimes \frac{\mathcal{O}_X}{\mathfrak{m}_x \cap \mathfrak{m}_y}\right)$$

**and**

$$H^0(X, L) \longrightarrow H^0(X, L \otimes \mathcal{O}_X/\mathfrak{m}_x^2)$$

**are surjective.** ■

**COROLLARY:** Let $L$ be a bundle on a compact complex manifold $X$, such that $H^1(X, L \otimes \mathcal{O}_X/\mathfrak{m}_x^2) = 0$ and $H^0\left(X, L \otimes \frac{\mathcal{O}_X}{\mathfrak{m}_x \cap \mathfrak{m}_y}\right) = 0$. **Then $L$ is very ample.** ■

## Very ample bundles on a curve

**THEOREM: (Kodaira-Nakano vanishing)**
Let $L$ be a line bundle on a compact complex manifold $X$, and $K_X$ its canonical bundle, Assume that $L \otimes K_X^{-1}$ is ample. **Then $H^i(L) = 0$ for all $i > 0$.**

**COROLLARY:** Let $L$ be a line bundle on a compact complex curve $C$ of genus $g$, and $\deg L > 2g$. **Then $L$ is very ample.**

**Proof. Step 1:** As shown above, **it suffices to show that $H^1(L \otimes_{\mathcal{O}_X} \otimes(\mathfrak{m}_x \cap \mathfrak{m}_y)) = 0$ and $H^1(L \otimes_{\mathcal{O}_X} (\mathfrak{m}_x^2) = 0$.** The sheaves $L \otimes_{\mathcal{O}_X} \otimes(\mathfrak{m}_x \cap \mathfrak{m}_y)$ and $L \otimes_{\mathcal{O}_X} (\mathfrak{m}_x^2)$ are line bundles of degree $\deg L - 2$ on $X$.

**Step 2:** The canonical bundle of $X$ has degree $2g-2$. By Kodaira, **a bundle on $X$ is ample if and only if it has positive degree.** By Kodaira-Nakano, $H^1(X, B) = 0$ **for any line bundle such that** $\deg(B \otimes K_X^{-1}) = \deg B - 2g + 2 > 0$.

**Step 3:** Comparing Step 1 and Step 2, we obtain that $L$ **is ample whenever** $\deg L - 2 - 2g + 2 > 0$. ■

## Canonical map for a complex curve

**DEFINITION:** Let $L$ be a line bundle on $X$. A point $p \in X$ is called **a base point of $L$** if all sections of $L$ vanish in $p$.

**DEFINITION:** Let $X$ be a complex manifold, such that the holomorphic sections of the canonical bundle have no common zeros. **The canonical map** is the standard map $X \longrightarrow \mathbb{P}H^0(X, K_X)^*$.

**THEOREM:** Let $C$ be a compact complex curve of genus $\geqslant 2$, and $K$ its canonical bundle. Then **the holomorphic sections of $K$ have no base points,** and **the canonical map $\Psi: C \longrightarrow \mathbb{P}H^0(K)$ is an embedding** or **a two-sheeted ramified covering with $\Psi(C) = \mathbb{C}P^1$**.

**REMARK:** In the second case $C$ is called **a hyperelliptic curve.** In step 3, we will prove that **any curve admitting a two-sheeted ramified covering to $\mathbb{C}P^1$ is hyperelliptic.**

## Canonical map for a complex curve

**Proof. Step 1:** We start by proving that **the sections of $K$ have no common zeros**. Take $p \in C$, let $k_p := \mathcal{O}_C/\mathfrak{m}_p$, and consider the exact sequence $0 \longrightarrow K(-p) \longrightarrow K \longrightarrow k_p \longrightarrow 0$. If $p$ is a base point, the corresponding long exact sequence implies that $H^1(K(-p)) \neq 0$. **Serre duality implies $H^1(K(-p)) = H^0(\mathcal{O}(p))^*$, hence there exists a rational section of $\mathcal{O}_C$ with a single pole**, that is, a holomorphic map to $\mathbb{C}P^1$ of degree 1, **This is impossible, unless $C = \mathbb{C}P^1$**, hence $K$ **has no base points.**

**Step 2:** If the canonical map glues together $p$ and $q \in C$, then $H^1(K(-p-q)) \neq 0$, giving, by Serre duality, $\dim H^0(\mathcal{O}(p+q)) \neq 0$. This implies that $C$ **admits a rational function with two poles**, that is, **a holomorphic map to $\mathbb{C}P^1$ of degree 2.**

**Step 3: It remains to show that $\Psi(C) = \mathbb{C}P^1$ if $C$ admits a two-sheeted ramified covering to $\mathbb{C}P^1$.** Let $\tau$ be the involution exchanging the sheets of the covering; by Riemann singularity removal theorem, **$\tau$ is holomorphic.** Since $\tau$ is an involution, it acts on the space $H^0(K_C)$ of holomorphic differentials with eigenvalues $\pm 1$. Since $\mathbb{C}P^1$ has no non-zero holomorphic differentials, $\tau$ acts on $H^0(K_C)$ as $-\operatorname{Id}$. **Therefore $\Psi$ glues $x$ and $\tau(x)$.** ■

## Finite morphisms

**DEFINITION:** Let $\Phi: X \longrightarrow Y$ be a morphism of complex varieties (or schemes). We say that $\Phi$ is **finite** if for any open set $U \subset Y$ the ring $\Phi^* \mathcal{O}_U$ is finite generated as $\mathcal{O}_V$-module, where $V = \Phi^{-1}(U)$.

**THEOREM:** A morphism $\Phi: X \longrightarrow Y$ of complex varieties (or algebraic varieties, or schemes of finite type) **is finite if and only if $\Phi$ is proper, and preimage of any point is finite.**

**Proof:** EGA IV, part 3, 8.11.1; Hartshorne, III, Exercise 11.2, `http://verbit.ru/IMPA/CV-2023/slides-cv-25.pdf`, Proposition 2. ■

## Ampleness and cohomology

**THEOREM:** Let $L$ be a line bundle on a scheme (or on a complex variety) $X$. **Then the $L$ is ample if and only if or any coherent sheaf $F$, there exists $d > 0$ such that $H^i(F \otimes L^{\otimes k}) = 0$ for all $i > 0$ and all $k \geqslant d$.**

**Proof:** See e.g. Hartshorne. ■

**THEOREM:** Let $f: X \longrightarrow Y$ be a finite morphism, and $F$ a coherent sheaf on $Y$. **Then $H^i(f^{-1}(U), F) = H^i(U, f_* F)$ for any open set $U \subset Y$; in other words, $R^i f_* f = 0$ for all $i > 0$.**

**Proof:** Ravi Vakil "Rising sea", Theorem 18.7.5. ■

**Corollary 1:** Let $L$ be a line bundle on a complex variety $X$ such that the standard map $f: X \longrightarrow \mathbb{P}H^0(X, L)^*$ is finite. **Then $L$ is ample.**

**Proof:** Let $Y = f(X)$, and $F$ a sheaf on $X$. By definition, $L = f^*(\mathcal{O}(1))$, hence $f_*(F \otimes_{\mathcal{O}_X} L^{\otimes k}) = f_* F \otimes_{\mathcal{O}_Y} \mathcal{O}(k)$ Then $H^i(X, F \otimes_{\mathcal{O}_X} L^{\otimes k}) = H^i(Y, f_*(F) \otimes_{\mathcal{O}_Y} \mathcal{O}(k))$, and this group vanishes for $k \gg 0$. ■

# Nakai-Moishezon theorem

**THEOREM: (Nakai-Moishezon)**
Let $X$ be a compact projective variety, and $L$ a line bundle on $X$. Assume that for any subvariety $Z \subset X$, $\dim Z = d$, one has $\int_Z c_1(L)^d > 0$. **Then $L$ is ample.**

**Proof. Step 1:** When $\dim X = 1$, one has $\operatorname{Pic}(X) = \mathbb{Z}$, and this statement is clear. We will prove Nakai-Moishezon using induction in $\dim X$. By inductive assumption we can assume that **Nakai-Moishezon is true for any projective variety $X_1 \subsetneq X$. Our next goal is to prove that $\lim_k \dim H^0(L^{\otimes k}) = \infty$.**

**Step 2:** Choose a very ample bundle $L_1$ with sufficiently big $c_1$ in such a way that $c_1(L \otimes L_1 \otimes K_X^{-1})$ is ample, and hence $H^i(L_1 \otimes L) = 0$ for $i > 0$. Let $H$ be a smooth zero divisor of a section of $L_1$, such that $L_1 = \mathcal{O}(H)$. We are going to show that the higher cohomology of $L^{\otimes d} \otimes \mathcal{O}(H)|_H$ vanishes for $d \gg 0$. This would follow if we prove that $L^{\otimes d} \otimes \mathcal{O}(H)|_H \otimes K_H^{-1}$ is ample.

Step 1 implies that $L|_H$ is ample. By adjunction formula, $K_H = K_M|_H \otimes \mathcal{O}(H)$, hence $K_H^{-1} = K_M^{-1}|_H \mathcal{O}(-H)$. Then $L^{\otimes d} \otimes \mathcal{O}(H)|_H \otimes K_H^{-1} = L^{\otimes d} \otimes K_M|_H$. Choosing $d$ sufficiently big, we may assume that $L^{\otimes d} \otimes \mathcal{O}(H)|_H \otimes K_H^{-1} = L^{\otimes d} \otimes K_M|_H$ is ample (Step 1), and hence the higher cohomology of $L^{\otimes d} \otimes \mathcal{O}(H)|_H$ vanishes. The same argument **shows that the higher cohomology of $L^{\otimes d} \otimes L^{\otimes q}|_H$ vanishes for all $q > 0$.**

## Nakai-Moishezon theorem (2)

**THEOREM: (Nakai-Moishezon)**
Let $X$ be a compact projective variety, and $L$ a line bundle on $X$. Assume that for any subvariety $Z \subset X$, $\dim Z = d$, one has $\int_Z c_1(L)^d > 0$. **Then $L$ is ample.**

**Step 2 - recall:** Let $L_1$ be an ample bundle on $X$, and $H$ a smooth zero divizor of its section. **In step 2, we have shown that the higher cohomology of $L^{\otimes d} \otimes L^{\otimes q}|_H$ vanishes for all $q > 0$.**

**Step 3:** Let $p \geqslant d$, where $d$ is chosen in Step 2. Consider the exact sequence

$$0 \longrightarrow L^{\otimes p} \otimes L_1^{\otimes q} \longrightarrow L^{\otimes p} \otimes L_1^{\otimes q+1} \longrightarrow L^{\otimes p} \otimes L_1^{\otimes q+1}|_H \longrightarrow 0.$$

The corresponding long exact sequence gives

$$H^{i-1}(L^{\otimes p} \otimes L_1^{\otimes q+1}|_H) \longrightarrow H^i(L^{\otimes p} \otimes L_1^{\otimes q}) \longrightarrow H^i(L^{\otimes p} \otimes L_1^{\otimes q+1}). \quad (*)$$

Fix $i \geqslant 2$. The cohomology group $H^{i-1}(L^{\otimes p} \otimes L_1^{\otimes q+1}|_H)$ vanishes for all $q \geqslant 0$ by Step 2. For $q$ sufficiently big, $H^i(L^{\otimes p} \otimes L_1^{\otimes q+1})$ also vanishes, because $L_1$ is ample. The exact sequence (*) implies that $H^i(L^{\otimes p} \otimes L_1^{\otimes q}) = 0$. Therefore, vanishing of $H^i(L^{\otimes p} \otimes L_1^{\otimes q+1})$ implies vanishing of $H^i(L^{\otimes p} \otimes L_1^{\otimes q})$ for any $q \geqslant 0$. This implies that $H^i(L^{\otimes p}) = 0$. **We have shown that $H^i(L^{\otimes p}) = 0$ for any $i \neq 0, 1$, if $p \geqslant d$, for some $d \gg 0$.**

## Nakai-Moishezon theorem (3)

**THEOREM: (Nakai-Moishezon)**
Let $X$ be a compact projective variety, and $L$ a line bundle on $X$. Assume that for any subvariety $Z\subset X$, $\dim Z=d$, one has $\int_Z c_1(L)^d>0$. **Then $L$ is ample.**

**Step 3, previous slide: There exists $d\gg 0$ such that $H^i(L^{d+j})=0$ for all $j\geqslant 0$, $i>1$.**

**Step 4:** Let $kL:=L^k=L^{\otimes k}$, Clearly, $c_1(kL)=kc_1(L)$. The Riemann-Roch-Hirzebruch formula gives $\chi(kL)=\int_X \exp(kc_1(L))\wedge td_*(TM)$, where

$$td_*=1+\frac{c_1}{2}+\frac{c_1^2+c_2}{12}+\frac{c_1c_2}{24}+\frac{-c_1^4+4c_1^2c_2+c_1c_3+3c_2^2-c_4}{720}+...$$

Let $\dim X=n$. Clearly, $\chi(kL)$ is polynomial in $k$. The degree $n$ term in $P(k):=\chi(kL)$ is $\int_X\frac{(kc_1(L))^n}{n!}$, by Riemann-Roch-Hirzebruch. This gives $\lim_k\chi(kL)=\infty$. Since $H^i(kL)=0$ for $k\gg 0$ and $i>1$, **this implies $\lim_k H^0(X,kL)=\infty$.**

## Nakai-Moishezon theorem (4)

**Steps 1-4:** By inductive assumption, we assume that Nakai-Moishezon is true for any projective variety $X_1 \subsetneq X$. **We have shown that $\lim_k H^0(X, kL) = \infty$.**

**Step 5:** Replacing $L$ by its power if necessary, we may assume that $\dim H^0(L) > 1$. Let $D$ be a zero divisor of a section of $L$. Consider the long exact sequence $0 \longrightarrow (k-1)L \longrightarrow kL \longrightarrow kL|_D \longrightarrow 0$. By inductive assumption, $L|_D$ is ample. Replacing $L$ by a sufficiently big power, we may assume that $H^i(kL|_D) = 0$ for all $k, i > 0$. **This gives a long exact sequence**

$$0 \longrightarrow H^0((k-1)L) \longrightarrow H^0(kL) \longrightarrow H^0(kL|_D) \longrightarrow H^1((k-1)L) \longrightarrow H^1(kL) \longrightarrow 0.$$

From this long exact sequence it is apparent the the function $k \mapsto \dim H^1(kL)$ is monotonous, hence stabilizes at some $k_0$. **Therefore, for all $k > k_0$, the map $H^0(kL) \longrightarrow H^0(kL|_D)$ is surjective.**

**Step 6:** Consider the birational map $\Phi: X \longrightarrow \mathbb{P}H^0(L)^*$. Then $D$ can be chosen as $\Phi^*(H)$, where $\Phi^*$ is the proper preimage, and $H$ a hyperplane section in $\mathbb{P}H^0(L)^*$. Therefore, for any two points $x, y \in X$, we may chose $D$ containing these two points. Since $L|_D$ is ample and $H^0(kL) \longrightarrow H^0(kL|_D)$ is surjective, **there exists a section of $kL$ separating these points** (vanishing in one and non-vanishing in the other).

**Step 7:** Now, **the bundle $kL$ is ample by Corollary 1.** ■