K3 surfaces

lecture 10: K3 surfaces with picard rank 1

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Very ample bundles (reminder)

DEFINITION: **Very ample bundle** on M is a line bundle obtained as $\varphi^*(\mathcal{O}(1))$, where $\varphi: M \hookrightarrow \mathbb{C}P^n$ is a projective embedding. **Ample bundle** is a line bundle L such that its tensor power $nL := L^{\otimes n}, n \geqslant 1$ is very ample.

THEOREM: **(Kodaira) A bundle L is ample if and only if $c_1(L)$ is a Kähler class.**

COROLLARY: **Any line bundle of positive degree on a compact complex curve is ample.**

COROLLARY: Let L be a bundle on a compact complex manifold X. Then the following are equivalent:

(i) **L is very ample**

(ii) for any two distinct points $x, y \in X$, **the natural maps** $H^0(X,L) \longrightarrow H^0\left(X, L \otimes \frac{\mathcal{O}_X}{\mathfrak{m}_x \cap \mathfrak{m}_y}\right)$ **and** $H^0(X,L) \longrightarrow H^0(X, L \otimes \mathcal{O}_X/\mathfrak{m}_x^2)$ **are surjective.** ■

Very ample bundles 2 (reminder)

COROLLARY: Let L be a bundle on a compact complex manifold X, such that $H^1(X, L \otimes \mathcal{O}_X/\mathfrak{m}_x^2) = 0$ and $H^0\left(X, L \otimes \frac{\mathcal{O}_X}{\mathfrak{m}_x \cap \mathfrak{m}_y}\right) = 0$. **Then L is very ample.** ■

THEOREM: (Kodaira-Nakano vanishing)
Let L be a line bundle on a compact complex manifold X, and K_X its canonical bundle, Assume that $L \otimes K_X^{-1}$ is ample. **Then $H^i(L) = 0$ for all $i > 0$.**

COROLLARY: Let L be a line bundle on a compact complex curve C of genus g, and $\deg L > 2g$. **Then L is very ample.**

Theorem 1: Let C be a compact complex curve of genus $\geqslant 2$, and K its canonical bundle. Then **the holomorphic sections of K have no base points,** and **the canonical map $\Psi: C \longrightarrow \mathbb{P}H^0(K)$ is an embedding** or **a two-sheeted ramified covering with $\Psi(C) = \mathbb{C}P^1$**.

REMARK: In the second case C is called **a hyperelliptic curve.**

CLAIM: Any curve admitting a two-sheeted ramified covering to $\mathbb{C}P^1$ is hyperelliptic.

Finite morphisms (reminder)

DEFINITION: Let $\Phi : X \longrightarrow Y$ be a morphism of complex varieties (or schemes). We say that Φ is **finite** if for any open set $U \subset Y$ the ring $\Phi^* \mathcal{O}_U$ is finite generated as $\mathcal{O}_V$-module, where $V = \Phi^{-1}(U)$.

THEOREM: A morphism $\Phi : X \longrightarrow Y$ of complex varieties (or algebraic varieties, or schemes of finite type) **is finite if and only if Φ is proper, and preimage of any point is finite.**

Proof: EGA IV, part 3, 8.11.1; Hartshorne, III, Exercise 11.2, `http://verbit.ru/IMPA/CV-2023/slides-cv-25.pdf`, Proposition 2. ■

Ampleness and cohomology (reminder)

THEOREM: Let L be a line bundle on a scheme (or on a complex variety) X. **Then the L is ample if and only if or any coherent sheaf F, there exists $d > 0$ such that $H^i(F \otimes L^{\otimes k}) = 0$ for all $i > 0$ and all $k \geqslant d$.**

Proof: See e.g. Hartshorne. ■

THEOREM: Let $f: X \longrightarrow Y$ be a finite morphism, and F a coherent sheaf on Y. **Then $H^i(f^{-1}(U), F) = H^i(U, f_* F)$ for any open set $U \subset Y$; in other words, $R^i f_* f = 0$ for all $i > 0$.**

Proof: Ravi Vakil "Rising sea", Theorem 18.7.5. ■

Corollary 1: Let L be a line bundle on a complex variety X such that the standard map $f: X \longrightarrow \mathbb{P}H^0(X, L)^*$ is finite. **Then L is ample.**

THEOREM: (Nakai-Moishezon)
Let X be a compact projective variety, and L a line bundle on X. Assume that for any subvariety $Z \subset X$, $\dim Z = d$, one has $\int_Z c_1(L)^d > 0$. **Then L is ample.**

Singular curve in a K3 surface

CLAIM: Let $C \subset M$ be a curve in a smooth complex surface. Then **there exists a surface $\tilde M \xrightarrow{\pi} M$ obtained by successive blow-ups of M such that the proper preimage $\tilde C$ of C is smooth.**

Proof: Let us blow up a singular point p of C, obtaining a resolution $M_1 \xrightarrow{\pi_1} M$, and let C be the proper preimage of C. Let $E \subset M_1$ be the exceptional divisor. The intersection index of C and a curve L passing through p is equal to the intersection index of $\tilde C + E$ with its proper preimage $\tilde L$, hence $(\tilde C, \tilde L) < (C, L)$. Therefore, **the multiplicity of the singularity of $\tilde C$ in the preimages of p is smaller than in p.** Using induction by multiplicity, we obtain that successive blow-ups resolve the singularity. ■

Singular curve in a K3 surface (2)

Let C be a curve in a K3 surface M, $\tilde{C}$ its proper preimage in a resolution $\tilde{M} \xrightarrow{\pi} M$ obtained by successive blowing up the singularities of C, and E the exceptional divisor of π. Denote by L the pullback of $\mathcal{O}(C)$ to $\tilde{M}$. Then $N(\tilde{C}) = \mathcal{O}(\tilde{C}) = L \otimes \mathcal{O}(-E)$ and $K_{\tilde{M}} = \mathcal{O}(E)$. **Adjunction formula implies $K_{\tilde{C}} = \mathcal{O}(E) \otimes L \otimes \mathcal{O}(-E) = L$.**

Corollary 2: Let C be a curve of genus > 0 in a K3 surface M. **Then $\mathcal{O}(C)|_C$ is globally generated.**

Proof: Since $\pi: \tilde{C} \longrightarrow C$ is finite, it would suffice to show that $\pi^*\mathcal{O}(C)$ is globally generated. However, $K_{\tilde{C}} = \pi^*\mathcal{O}(C)$ is globally generated by Theorem 1. ■

K3 surfaces with $\operatorname{Pic}(M) = \mathbb{Z}$: either L or L^* is globally generated

THEOREM: Let M be a K3 surface, such that $\operatorname{Pic}(M) = \mathbb{Z}$, and L the line bundle generating $\operatorname{Pic}(M)$. Assume that $(L, L) \geqslant 0$. **Then L or L^* is globally generated.**

Proof. Step 1: Riemann-Roch-Hirzebruch gives $h^0(L) - h^1(L) + h^2(L) = \chi(L) = 2 + \frac{(L,L)}{2}$, and Serre duality gives $H^0(L^*)^* = H^2(L \otimes K_M) = H^2(L)$, hence $h^0(L^*) = h^2(L)$. **Therefore, $h^0(L) + h^0(L^*) \geqslant 2$, hence either L or L^* have have non-zero holomorphic sections.** Replacing L by L^* if necessary, **we can assume that $h^0(L) > 1$.**

Step 2: Let D be the zero divisor of a a general section of L. Since $[D]$ generates $H^{1,1}(M) \cap H^2(M, \mathbb{Z}) = \operatorname{Pic}(M)$, the divisor D is irreducible. From the exact sequence $0 \longrightarrow \mathcal{O}_M \longrightarrow L \longrightarrow L|_D \longrightarrow 0$ and $H^1(\mathcal{O}_M) = 0$ it follows that **the restriction mao $\Psi : H^0(M, L) \longrightarrow H^0(D, L|_D)$ is surjective:** every section of $L|_D$ can be restricted to a section of L.

Step 3: The bundle $L|_D$ is base point free by Corollary 1. Since $h^0(L) > 1$, **the union of all zero divisors of sections of L is M;** then Step 2 implies that L is base point free. ■

Remark 1: Let $(\tilde{M}, \tilde{D}) \xrightarrow{\pi} (M, D)$ be a resolution of singularities of D. Since $\pi^* L = K_{\tilde{D}}$, **the restriction $\pi^* L\Big|_{\tilde{D}}$ is very ample if and only if D is not hyperelliptic.**

K3 surfaces with $\operatorname{Pic}(M) = \mathbb{Z}$: L or L^* is ample

DEFINITION: Let L be a line bundle. The **line system** defined by L is the set of all zero divisors of all holomorphic sections of L. We denote this set as $|L|$. The base set of L is $\bigcap_{D\in|L|} D$; L is **globally generated** if this set is empty.

THEOREM: Let M be a K3 surface, such that $\operatorname{Pic}(M) = \mathbb{Z}$, and L a line bundle generating $\operatorname{Pic}(M)$. Assume that $(L, L) > 2$. **Then L or L^* is ample, base point free, and the map $\Psi : M \longrightarrow \mathbb{P}H^0(M, L)^*$ is an embedding or a 2-sheeted ramified cover.**

Proof. Step 1: As usual, we replace L by L^* if it has no sections. Let $\Psi : M \longrightarrow \mathbb{P}H^0(M, L)^*$ be the standard map; **it is holomorphic as shown above.** Ψ does not contract curves, because $\operatorname{Pic}(M) = \mathbb{Z}$, hence $L = \Psi^*(\mathcal{O}(1))$ restricted to any curve is non-trivial. If Ψ glues together points $x \neq y$, any curve $D \in |L|$ passing through x and y is hyperelliptic (Remark 1).

Step 2: Every such D is obtained as a preimage of a hyperplane section containing $\Psi(x)$. The union of all such D is M. Therefore, **if $|L|$ contains at least one hyperelliptic curve, Ψ is 2-sheeted, and all curves $D \in |L|$ are hyperelliptic.**

Step 3: The bundle L is ample by Corollary 1. ■

Hyperelliptic curves

Lemma 1: Let C be a hyperellptic curve of genus g. **Then the hyperelliptic involution has $2g$ fixed points on C.**

Proof: Let f be the number of the fixed points, and $e(C)$ the Euler characteristic. Riemann-Hurwitz formula gives $2-2g = e(C) = 2e(\mathbb{C}P^1) - f = 2 - f$.
■

PROPOSITION: All curves of genus 2 are hyperelliptic.

Proof: Let C be a genus 2 curve. Serre duality implies $\chi(K_C) = -\chi(\mathcal{O}_C) = g-1$. Also Serre duality implies that $H^1(K_C) = \mathbb{C}$, which gives $\dim H^0(K_C) = \chi(K_C) + 1 = g = 2$. Since K_C is base point free, the natural map $\Phi : C \longrightarrow \mathbb{P}H^0(K_C)^*$ is a holomorphic map to $\mathbb{P}H^0(K_C)^* = \mathbb{C}P^1$. **This map cannot be an isomorphism, hence it is a 2-sheeted covering.** ■

K3 surfaces with Pic$(M) = \mathbb{Z}$: L or L^* is very ample if $(L, L) > 2$

THEOREM: Let M be a K3 surface, such that Pic$(M) = \mathbb{Z}$, and L the line bundle generating Pic(M). **Then the map $\Psi : M \longrightarrow \mathbb{P}H^0(M, L)^*$ is a 2-sheeted ramified cover if $(L, L) = 2$ and an embedding otherwise; in the first case, M is a 2-sheeted covering of $\mathbb{C}P^2$ ramified in a sextic.**

Proof. Step 1: As usual, we replace L by L^* if it has no sections. Since L is base point free, a general element D of $|L|$ is smooth by Bertini theorem. Since $K_D = L|_D$, the genus $g(D) = \frac{(L,L)}{2} + 1$. Let $R \subset M$ be the ramification divisor. A general D meets R in $2g(D)$ points by Lemma 1, hence $(L, L) + 2$ is divisible by (L, L). **Since (L, L) is even, this is possible only when $(L, L) = 2$.**

Step 2: If $(L, L) = 2$, the genus of all curves $D \in |L|$ is 2, hence they are hyperelliptic. Then $\Psi : M \longrightarrow \mathbb{P}H^0(M, L)^*$ is a 2-sheeted cover. Since L is ample, $H^1(L) = 0$, hence the Riemann-Roch formula gives $\chi(L) = 2 + \frac{(L,L}{2} = 3 = \dim H^0(L)$. **Then $\mathbb{P}H^0(M, L)^*$ is 2-dimensional, and $\Psi : M \longrightarrow \mathbb{P}H^0(M, L)^*$ is a ramified cover.**

K3 surfaces with $\operatorname{Pic}(M) = \mathbb{Z}$: L or L^* is ramified in a sextic if $(L, L) = 2$

Step 3: It remains to show that Ψ is ramified in a sextic. Let $R \subset M$ be the ramification divisor. Since R is the fixed set of a holomorphic involution, it is smooth. Then $K_M = \Psi^* K_{\mathbb{C}P^2} \otimes \mathcal{O}(R) = \mathcal{O}_M$, and the self-intersection of $c_1(\Psi^* K_{\mathbb{C}P^2})$ is 18, because $c_1(K_{\mathbb{C}P^2})^2 = 9$. This implies that $\Psi^* K_{\mathbb{C}P^2} = L^{\otimes 3}$ and $[R] = 3c_1(L)$. Let $R_0 = \Psi(R)$. The intersection of R_0 with a transversal hyperplane section has the same number of points as $R \cap D = 6$, hence R_0 is a sextic. ■

REMARK: Converse is also true, by the same formula $K_M = \Psi^* K_{\mathbb{C}P^2} \otimes \mathcal{O}(R) = \mathcal{O}_M$: **a double cover of $\mathbb{C}P^2$ ramified in sextic is a K3.**

Corollary 3: Let M be a K3 surface, such that $\operatorname{Pic}(M) = \mathbb{Z}$, and L the line bundle generating $\operatorname{Pic}(M)$. Assume that $(L, L) > 2$. **Then L or L^* is very ample.** ■

Ample bundles on quartic surfaces

PROPOSITION: A K3 surface M is isomorphic to a quartic **if and only if** Pic(M) **contains a very ample bundle L with $(L,L)=4$.**

Proof. Step 1: Suppose that M is isomorphic to a quartic, let $\Phi: \ M \longrightarrow \mathbb{C}P^3$ be the projective embedding, and $L := \Phi^*(\mathcal{O}(1))$. Then $(L,L) = \int_M c_1(L)\wedge c_1(L) = \int_{\mathbb{C}P^3}[M]\wedge[H]\wedge[H]$, where $[H]$ is the fundamental class of a hyperplane section. **Since M is a quartic, $[M]=4[H]$, which gives $(L,L)=4$.**

Step 2: Conversely, let L be a very ample bundle on a K3 such that $(L,L)=4$. Riemann-Roch-Hirzebruch give $h^0(L) = h^0(L) - h^1(L) + h^2(L) = \chi(L) = 2 + \frac{(L,L)}{2} = 4$. **The corresponding embedding $M \longrightarrow \mathbb{P}H^0(M,L)^*$ takes M to a hypersurface of degree $\int_{\mathbb{C}P^3}[M]\wedge[H]\wedge[H] = (L,L) = 4$.** ■

COROLLARY: Let M be a K3 surface, such that Pic$(M)=\mathbb{Z}$, and L the line bundle generating Pic(M). Assume that $(L,L)=4$. **Then M is isomorphic to a quartic.**

Proof: It is very ample by Corollary 3, hence by the previous proposition M is a quartic. ■