K3 surfaces

lecture 11: Density of quartics deduced from Ratner theory

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Ample bundles on quartic surfaces (reminder from lecture 10)

PROPOSITION: A K3 surface M is isomorphic to a quartic **if and only if** Pic(M) **contains a very ample bundle L with $(L, L) = 4$.**

COROLLARY: Let M be a K3 surface, such that Pic$(M) = \mathbb{Z}$, and L the line bundle generating Pic(M). Assume that $(L, L) > 2$. **Then L or L^* is very ample.** ■

Corollary 1: Let M be a K3 surface, such that Pic$(M) = \mathbb{Z}$, and L the line bundle generating Pic(M). Assume that $(L, L) = 4$. **Then M is isomorphic to a quartic.**

Proof: The bundle L is very ample by the previous corollary, hence by the previous proposition M is a quartic. ■

REMARK: Today we shall prove that **quartics are dense in the Teichmüller space of K3 surfaces,** using local Torelli theorem (which will be proven later).

The period space of complex structures (reminder from lecture 7)

DEFINITION: Let Teich be the Teichmüller space of complex structures of Kähler type on a K3 surface. The corresponding period space is denoted $\mathbb{P}\mathrm{er} := \{v \in \mathbb{P}H^2(M,\mathbb{C}) \quad | \quad \int_M v \wedge v = 0, \int_M v \wedge \overline{v} > 0\}$.

PROPOSITION: (local Torelli theorem)
Let Teich be the space of complex structures on a K3 surface, and Per : Teich $\longrightarrow \mathbb{P}\mathrm{er}$ the map taking (M,I) to the line $H^{2,0}(M) \subset H^2(M,\mathbb{C})$. **Then Per is a local diffeomorphism.**
Proof: Later in these lectures.

CLAIM: $\mathbb{P}\mathrm{er} = SO(b_2-3,3)/SO(2) \times SO(b_2-3,1)$.
Proof: Take a non-zero vector v in a line $l \in \mathbb{P}\mathrm{er}$. Since $(v,v) = 0$ and $(v,\overline{v}) > 0$, the vectors v and $\overline{v}$ are not proportional, hence they generate a 2-dimensional plane $P \subset H^2(M,\mathbb{R})$ which is positive, because $(v,\overline{v}) > 0$, hence belongs to the positive, oriented Grassmannian

$$\mathrm{Gr}_{++}(H^2(M,\mathbb{R})) = SO(b_2-3,3)/SO(2) \times SO(b_2-3,1).$$

Conversely, for any $P \subset \mathrm{Gr}_{++}(H^2(M,\mathbb{R}))$, its complexification $P \otimes \mathbb{C}$ contains two lines l_1, l_2 which belong to the quadric $q(v,v) = 0$. These two lines are distinguished by their orientation. This implies that **the correspondence $\mathbb{P}\mathrm{er} \to SO(b_2-3,3)/SO(2) \times SO(b_2-3,1)$ taking $\Omega \in Q$ to $\langle \mathrm{Re}\,\omega, \mathrm{Im}\,\Omega\rangle \in \mathrm{Gr}_{++}(H^2(M,\mathbb{R}))$ is bijective.** ■

Noether-Lefschetz locus in the period space

DEFINITION: Let M be a K3, $\eta \in H^2(M,\mathbb{R})$ be a non-zero class, and $\mathbb{P}\mathrm{er}_\eta \subset \mathbb{P}\mathrm{er} = \{v \in \mathbb{P}H^2(M,\mathbb{C}) \mid \int_M v \wedge v = 0, \int_M v \wedge \overline{v} > 0\}$. the set of all points such that $\eta \perp v$, **or, equivalently, $\eta \in H^{1,1}(M,I)$ for any $I \in$ Teich such that $\mathrm{Per}(I) = v$.** Then $\mathbb{P}\mathrm{er}_\eta$ is called **the Noether-Lefschetz locus corresponding to η.**

REMARK: Clearly, $\mathbb{P}\mathrm{er}_\eta$ is an intersection of a complex hyperplane $\mathbb{P}\eta^\perp \subset \mathbb{P}H^2(M,\mathbb{C})$ and $\mathbb{P}\mathrm{er}$, hence **$\mathbb{P}\mathrm{er}_\eta$ is a complex divisor in $\mathbb{P}\mathrm{er}$.** This divisor is clearly smooth **(all homogeneous varieties are smooth).**

REMARK: For the same reason, given any subspace $V \subset H^2(M,\mathbb{R})$, the period space $\mathbb{P}\mathrm{er}(V) := \{v \in \mathbb{P}V \otimes_{\mathbb{R}} \mathbb{C} \mid \int_M v \wedge v = 0, \int_M v \wedge \overline{v} > 0\}$ is a complex submanifold of $\mathbb{P}\mathrm{er}$, with $\dim_{\mathbb{C}} \mathbb{P}\mathrm{er}(V) = 2\dim_{\mathbb{R}} M - 2$. However, **$\mathbb{P}\mathrm{er}(V)$ is empty if V does not contain positive 2-planes.** The submanifold $\mathbb{P}\mathrm{er}(V)$ is also called **the Noether-Lefschetz locus.**

The set of quartics with Picard rank 1

REMARK: Given $v \in \mathbb{P}\mathrm{er}$, let $H^{1,1}(M,v)$ be the space $\langle \operatorname{Re} v, \operatorname{Im} v\rangle^\perp \subset H^2(M,\mathbb{R})$. **Clearly, $H^{1,1}(M,v) = H^{1,1}(M,I)$ whenever $v = \operatorname{Per}(I)$.** We denote by $\operatorname{Pic}(M,v)$ the lattice $H^{1,1}(M,v) \cap H^2(M,\mathbb{Z})$; **clearly, it coincides with $\operatorname{Pic}(M,I)$ whenever $v = \operatorname{Per}(I)$.**

CLAIM: Let $\eta \in H^2(M,\mathbb{Z})$ and $\mathbb{P}\mathrm{er}_\eta^0$ be the space of all $v \in \mathbb{P}\mathrm{er}_\eta$ such that $\operatorname{rk}\operatorname{Pic}(M,v) = 1$. Then **$\mathbb{P}\mathrm{er}_\eta^0$ is dense in $\mathbb{P}\mathrm{er}_\eta$.**

Proof: Let $\mathfrak{S}$ be the set of all rank 2 subgroups of $H^2(M,\mathbb{Z})$ containing η. **Then $\mathbb{P}\mathrm{er}(V) \subset \mathbb{P}\mathrm{er}_\eta$ is a smooth divisor in $\mathbb{P}\mathrm{er}_\eta$.** Since

$$\mathbb{P}\mathrm{er}_\eta \backslash \mathbb{P}\mathrm{er}_\eta^0 = \bigcup_{S \in \mathfrak{S}} \mathbb{P}\mathrm{er}(V)$$

the complement $\mathbb{P}\mathrm{er}_\eta \backslash \mathbb{P}\mathrm{er}_\eta^0$ is a countable union of divisors, hence it has measure 0. ∎

Set of all quartics is dense

THEOREM: Let M be a K3 and $\mathfrak{R} \subset H^2(M,\mathbb{Z})$ the set of all vectors η such that $(\eta,\eta) = 4$. **Then $\bigcup_{\eta\in\mathfrak{R}} \mathbb{P}\mathrm{er}_\eta$ is dense $\mathbb{P}\mathrm{er}$.**

Proof: Later today.

Using this result and local Torelli, we prove

Corollary 2: Let $\mathfrak{Q} \subset$ Teich be the set of all K3 with Picard group of rank 1 generated by a vector x with $(x,x) = 4$ **Then $\mathfrak{Q}$ is dense in Teich.**

Proof: In a neighbourhood of each point $v \in \mathbb{P}\mathrm{er}$ there is a point $w \in \mathbb{P}\mathrm{er}_\eta$, where $\eta \in \mathbb{R}$. Since w is a limit of points $w_i \in \mathbb{P}\mathrm{er}_\eta^0 \subset \mathfrak{Q}$, every neighbourhood of v contains a point in $\mathfrak{Q}$. ■

REMARK: By Corollary 1, **all such x correspond to quartics;** therefore, **Corollary 2 implies that quartics are dense in Teich.**

COROLLARY: Every K3 is diffeomorphic to a smooth quartic. ■

Set of all quartics is dense (2)

It remains to prove

THEOREM: Let M be a K3 and $\mathfrak{R} \subset H^2(M,\mathbb{Z})$ the set of all vectors η such that $(\eta,\eta) = 4$. **Then $\bigcup_{\eta\in\mathfrak{R}} \mathbb{P}\mathrm{er}_\eta$ is dense $\mathbb{P}\mathrm{er}$.**

Using the identification $\mathbb{P}\mathrm{er} = \mathrm{Gr}_{++}(H^2(M,\mathbb{R}))$ we reformulate this theorem as follows.

Theorem 2: Let M be a K3, $\mathfrak{R} \subset H^2(M,\mathbb{Z})$ the set of all vectors η such that $(\eta,\eta) = 4$, and $W_\mathfrak{R} \subset \mathrm{Gr}_{+,+}(H^2(M,\mathbb{R}))$ the set of all 2-planes orthogonal to some $\eta \in \mathfrak{R}$. **Then $W_\mathfrak{R}$ is dense in $\mathrm{Gr}_{+,+}(H^2(M,\mathbb{R}))$.**

REMARK: There are 2 proofs. Today we will deduce Theorem 2 from Ratner's theorems (a deep and fundamental result of homogeneous dynamics). Next lecture we prove it directly using an elementary argument.

Ergodic measures

DEFINITION: Let (M,μ) be a space with a measure, and G a group acting on M preserving μ. This action is **ergodic** if all G-invariant measurable subsets $M' \subset M$ satisfy $\mu(M') = 0$ or $\mu(M\backslash M') = 0$.

REMARK: Ergodic measures are extremal rays in the cone of all G-invariant measures.

REMARK: By Choquet's theorem, **any G-invariant measure on M is expressed as an average of a certain set of ergodic measures.** Therefore, **G-invariant ergodic measures always exist.**

CLAIM: Let M be a manifold, μ a Lebesgue measure, and G a group acting on (M,μ) ergodically. **Then the set of non-dense orbits has measure 0.**

Proof: Consider a non-empty open subset $U \subset M$. Then $\mu(U) > 0$, hence $M' := G \cdot U$ satisfies $\mu(M\backslash M') = 0$. For any orbit $G \cdot x$ not intersecting U, $x \in M\backslash M'$. Therefore the set of such orbits has measure 0. ■

Ratner theory

DEFINITION: Let G be a connected Lie group equipped with a Haar measure. **A lattice** $\Gamma \subset G$ is a discrete subgroup of finite covolume (that is, G/Γ has finite volume).

EXAMPLE: By Borel and Harish-Chandra theorem, **any integer lattice in a simple Lie group has finite covolume.**

THEOREM: (Calvin C. Moore, 1966) Let Γ be an arithmetic lattice in a non-compact simple Lie group G with finite center, and $H \subset G$ a non-compact subgroup. Then the left action of Γ on G/H is **ergodic,** that is, **for all Γ-invariant measurable subsets $Z \subset G/H$, either Z has measure 0, or $G/H \backslash Z$ has measure 0.**

THEOREM: (Marina Ratner)
Let $H \subset G$ be a Lie subroup generated by unipotents, and $\Gamma \subset G$ a lattice. Then **a closure of any H-orbit in G/Γ is an orbit of a closed, connected subgroup $S \subset G$, such that $S \cap \Gamma$ is a lattice in S.**

REMARK: Let $x \in G/H$ be a point in a homogeneous space, and $\Gamma \cdot x$ its Γ-orbit, where Γ is an arithmetic lattice. Then its closure is an orbit of a group S containing stabilizer of x. Moreover, **S is a smallest group defined over rationals and stabilizing x.**

Oppenheim conjecture

CONJECTURE: (Oppenheim, 1929; proven by G. Margulis, 1987) Let q be an irrational quadratic form on $\mathbb{R}^n$ of signature (a,b), with $a,b>0$, and $S_q := q(\mathbb{Z}^n)$. **Then S is dense in $\mathbb{R}$.**

Proof. Step 1: Let $G = SL(n,\mathbb{R})$ and $H = SO(a,b) \subset G$, and $\Gamma = SL(n,\mathbb{Z})$. Points of G/H classify quadratic forms of signature (a,b). Consider the function $F: G/H \longrightarrow \mathbb{R}$ taking a frame $e_1, ..., e_n$ to $q(e_1)$. Clearly, $F(\Gamma\cdot(e_1, ..., e_n)) \subset S_q$ when e_1 is integer, hence **it suffices to show that the orbit $\Gamma\cdot q$ is dense**.

Step 2: There are no intermediate subgroups between $SO(a,b)$ and $SL(n,\mathbb{R})$ **(an exercise).** Then, by Ratner's theorem, a point $q \in G/H$ has a closed orbit (and in this case q is preserved by a sublattice $H_\mathbb{Z} = G_\mathbb{Z} \cap H$; in other words, q is a rational quadratic form), or a dense orbit, and in this case S_q is dense. ■

Orbits of $SO(H^2(M,\mathbb{Z}))$ on $\mathrm{Gr}_{+,+}(H^2(M,\mathbb{R}))$

EXERCISE: Let $G = SO(a,b)$ and $H \subset G$ the stabilizer of a point $W \in \mathrm{Gr}_{++}(\mathbb{R}^{a,b})$. **Then there is only one type of intermediate subgroups between G and H: it is the stabilizer of a non-zero vector $x \in W$.**

Therefore, Ratner theorem implies
PROPOSITION: Let $v \in \mathbb{P}\mathrm{er}$ be a point corresponding to a 2-plane $W \in \mathrm{Gr}_{++}(H^2(M,\mathbb{R}))$ with no rational vectors. **Then its $SO(H^2(M,\mathbb{Z}))$-orbit in $\mathrm{Gr}_{++}(H^2(M,\mathbb{R}))$ is dense.**

Now, Theorem 2 **easily follows from this proposition,** because a general point $v \in \mathbb{P}\mathrm{er}_\eta$ has no rational vectors.

Marina Ratner (1979).