K3 surfaces

lecture 12: Density of quartics, a more elementary proof

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Ample bundles on quartic surfaces (reminder from lecture 10)

PROPOSITION: A K3 surface M is isomorphic to a quartic **if and only if** $\operatorname{Pic}(M)$ **contains a very ample bundle L with $(L,L)=4$.**

COROLLARY: Let M be a K3 surface, such that $\operatorname{Pic}(M)=\mathbb{Z}$, and L the line bundle generating $\operatorname{Pic}(M)$. Assume that $(L,L)>2$. **Then L or L^* is very ample.** ■

Corollary 1: Let M be a K3 surface, such that $\operatorname{Pic}(M)=\mathbb{Z}$, and L the line bundle generating $\operatorname{Pic}(M)$. Assume that $(L,L)=4$. **Then M is isomorphic to a quartic.**

Proof: The bundle L is very ample by the previous corollary, hence by the previous proposition M is a quartic. ■

In Lecture 11, we deduced density of quartics from Torelli theorem, and the following result.

Theorem 2: Let M be a K3, $\mathfrak{R}\subset H^2(M,\mathbb{Z})$ the set of all vectors η such that $(\eta,\eta)=4$, and $Z(\mathfrak{R})\subset \operatorname{Gr}_{+,+}(H^2(M,\mathbb{R}))$ the set of all 2-planes orthogonal to some $\eta\in\mathfrak{R}$. **Then $Z(\mathfrak{R})$ is dense in $\operatorname{Gr}_{+,+}(H^2(M,\mathbb{R}))$.**

REMARK: There are 2 proofs. In Lecture 11 we deduced Theorem 2 from Ratner's theorems (a deep and fundamental result of homogeneous dynamics). Today we will prove it directly using a more elementary argument.

Quadratic lattices

We will prove a more general result about quadraic lattices. Recall that **a quadratic lattice** is $\mathbb{Z}^n$ equipped with an integer-valued quadratic form.

DEFINITION: Let k be an integer. We say that an integer quadratic lattice (Λ, q) **represents** k if there exists $u \in \Lambda$ such that $q(u,u) = k$.

Let $(V_{\mathbb{Z}}, q)$ be a non-degenerate quadratic lattice of signature (a,b) with $a \geqslant 3, b \geqslant 1$, and $g \in \mathbb{Z}$ a number such that there exists $x \in V_{\mathbb{Z}}$ such that $q(x,x) \neq 0$. Denote by $V_{\mathbb{R}}$ the tensor product $V_{\mathbb{R}} := V_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{R}$.

We can now forget about K3 and consider Theorem 2 as a special case of the following

Theorem 1: Let $\mathfrak{R} \subset V_{\mathbb{Z}}$ the set of all vectors η such that $(\eta,\eta) = g$, where g is a positive integer represented by q, and $Z(\mathfrak{R}) \subset \operatorname{Gr}_{+,+}(V_{\mathbb{R}})$ the set of all 2-planes orthogonal to some $\eta \in \mathfrak{R}$. **Then $Z(\mathfrak{R})$ is dense in $\operatorname{Gr}_{+,+}(V_{\mathbb{R}})$.**

Its proof takes the rest of today's lecture.

Geometric idea of the proof: Let $\Gamma := SO(V_{\mathbb{Z}}, q)$. Clearly, $\mathfrak{R}$ coincides with its Γ-orbit. Consider the set $\operatorname{Null}(V) \subset \mathbb{P}(V_{\mathbb{R}})$ of all vectors x such that $q(x,x) = 0$. It is well known that $\operatorname{Null}(V)$ belongs to the closure of an orbit $\Gamma \cdot x$ for any $x \in \mathbb{P}(V_{\mathbb{R}})$. **We will prove that the closure of $Z(\mathfrak{R})$ coincides with Z(its closure in $\mathbb{P}(V_{\mathbb{R}})$), which contains $Z(\operatorname{Null}(V)) = \operatorname{Gr}_{+,+}(V_{\mathbb{R}})$.**

Closures and the sets of orthogonal 2-planes

Let $A \subset \mathbb{P}V_{\mathbb{R}}$ be a subset. Denote by $Z(A) \subset \operatorname{Gr}_{++}(V_{\mathbb{R}})$ the set of all 2-planes orthogonal to some $a \in A$.

CLAIM: Let $U \subset \operatorname{Gr}_{++}(V_{\mathbb{R}})$ be an open subset, and $P(U)$ be the set of all vectors $v \in V_{\mathbb{R}}$ such that $v \perp W$ for some $W \in U$. **Then $P(U)$ is open in $V_{\mathbb{R}}$.**

Proof: Left as an exercise. ■

Lemma 1: Let $B \subset \mathbb{P}V_{\mathbb{R}}$ be the closure of $A \subset \mathbb{P}V_{\mathbb{R}}$. **Then $Z(B)$ is the closure of $Z(A)$.**

Proof. Step 1: Let $\{a_i\} \subset \mathbb{P}V_{\mathbb{R}}$ be a sequence, $\{W_i \in Z(a_i)\}$ a sequence of 2-planes orthogonal to a_i, and $W \in \operatorname{Gr}_{++}(V_{\mathbb{R}})$ its limit. Since $\mathbb{P}V_{\mathbb{R}}$ is compact, we can replace $\{a_i\}$ by its subsequence which converges to $b \in \mathbb{P}V_{\mathbb{R}}$. **Since the scalar product is continuous, $W \perp b$, hence $\overline{Z(A)} \subset Z(B)$.**

Step 2: Conversely, let $\{a_i\} \subset \mathbb{P}V_{\mathbb{R}}$ be a sequence which converges to $b \in \mathbb{P}V_{\mathbb{R}}$, and $W \in Z(b)$. Given an open subset $U \ni W$ in $\operatorname{Gr}_{++}(V_{\mathbb{R}})$, the set $\{x \in \mathbb{P}V_{\mathbb{R}} \mid Z(x) \cap U \neq \emptyset\}$ is open by the previous claim, **hence it contains almost all $\{a_i\}$.** Therefore, W is a limit of a sequence $W_i \in Z(a_i)$. This implies that $Z(B) \subset \overline{Z(A)}$. ■

The null-quadric

DEFINITION: The light cone, or **null-quadric** $\operatorname{Null}(V) \subset \mathbb{P}V_{\mathbb{R}}$ is the set $\{l \in \mathbb{P}V_{\mathbb{R}},\ (l,l) = 0\}$.

REMARK: $Z(\operatorname{Null}(V_{\mathbb{R}})) = \operatorname{Gr}_{+,+}(V_{\mathbb{R}})$. Indeed, for any positive 2-plane in $V_{\mathbb{R}}$, its orthogonal complement contains a null-vector, because the signature of V is at least $(3,1)$.

REMARK: To prove Theorem 1, we will show that $\operatorname{Null}(V) \subset \overline{\mathfrak{R}}$. Then Lemma 1 gives $\overline{Z(\mathfrak{R})} \supset Z(\operatorname{Null}(V)) = \operatorname{Gr}_{+,+}(V_{\mathbb{R}})$

We have just reduced Theorem 1 to Theorem 3 below.

Theorem 1: Let $\mathfrak{R} \subset V_{\mathbb{Z}}$ the set of all vectors η such that $(\eta,\eta) = g$, and $Z(\mathfrak{R}) \subset \operatorname{Gr}_{+,+}(V_{\mathbb{R}})$ the set of all 2-planes orthogonal to some $\eta \in \mathfrak{R}$. **Then $Z(\mathfrak{R})$ is dense in $\operatorname{Gr}_{+,+}(V_{\mathbb{R}})$.**

Theorem 3: Let $\mathfrak{R} \subset V_{\mathbb{Z}}$ the set of all vectors η such that $(\eta,\eta) = g$. **Then the closure of $\mathbb{P}\mathfrak{R} \subset \mathbb{P}V_{\mathbb{R}}$ contains $\operatorname{Null}(V_{\mathbb{R}})$.**

Proof: Next lecture.