K3 surfaces

lecture 13: Limit points of orbits of $SO_{\mathbb{Z}}(p,q)$

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Summary from the last lecture

Let $(V_{\mathbb Z}, q)$ be a non-degenerate quadratic lattice of signature (a, b) with $a \geqslant 3, b \geqslant 1$, and $g \in \mathbb Z$ a number such that there exists $x \in V_{\mathbb Z}$ such that $q(x, x) \neq 0$. Denote by $V_{\mathbb R}$ the tensor product $V_{\mathbb R} := V_{\mathbb Z} \otimes_{\mathbb Z} \mathbb R$. As usual, we denote the Grassmannian of positive, oriented 2-planes by $\operatorname{Gr}_{+,+}(V_{\mathbb R})$ and the null-quadriv $\{l \in \mathbb P(V_{\mathbb R}) \ \mid \ q(l, l) = 0\}$ by $\operatorname{Null}(V_{\mathbb R})$.

In lecture 12, **we reduced density of the quartics in the Teichmüller space of K3 surfaces to the following statement.**

Theorem 1: Let $\mathfrak R \subset V_{\mathbb Z}$ the set of all vectors η such that $(\eta, \eta) = g$, and $Z(\mathfrak R) \subset \operatorname{Gr}_{+,+}(V_{\mathbb R})$ the set of all 2-planes orthogonal to some $\eta \in \mathfrak R$. **Then $Z(\mathfrak R)$ is dense in $\operatorname{Gr}_{+,+}(V_{\mathbb R})$.**

and reduced it further to
Theorem 3: Let $\mathfrak R \subset V_{\mathbb Z}$ the set of all vectors η such that $(\eta, \eta) = g$. **Then the closure of $\mathbb P\mathfrak R \subset \mathbb P V_{\mathbb R}$ contains $\operatorname{Null}(V_{\mathbb R})$.**

Also, **we proved existence of solutions of Pelle's equation**

THEOREM: (Lagrange)
Let $w > 1$ be a square-free integer. **Then the equation $x^2 - y^2 w = 1$ has non-trivial integer solutions.**

Commensurability (reminder)

DEFINITION: Two subgroups $G_1, G_2 \subset GL(n, \mathbb{R})$ are called **commensurable** if $G_1 \cap G_2$ has finite index in G_1 and in G_2.

PROPOSITION: Let $\Lambda_1 \subset \Lambda$ be quadratic lattices of the same rank. **Then $SO(\Lambda_1)$ and $SO(\Lambda)$ are commensurable.**

Corollary 2: Let (B, q) be a non-degenerate quadratic lattice, and $A \subset B$ a non-degenerate sublatice of smaller rank. Denote by $\Gamma_A \subset SO(A)$ the group of all isometries of A which can be extended to an isometry of B. **Then Γ_A is of finite index in $SO(A)$.**

Pelle's equation (reminder)

REMARK: Let w be a square-free integer. Consider $\mathcal{O}_K := \mathbb{Z} + \mathbb{Z}w$, let σ be its automorphism $\sigma(a+b\sqrt{w}) := a - \sqrt{w}b$. Denote by $N(a+b\sqrt{w})$ the number $N(a+b\sqrt{w}) = a^2 - wb^2(a+b\sqrt{w})\sigma(a+b\sqrt{w})$ Clearly, the norm $N(z) = z\sigma(z)$ is multiplicative on $\mathcal{O}_K$, **the product of solutions of an equation $N(z) = 1$ is another solution.** Also, solutions of $N(z) = 1$ are invertible in $\mathcal{O}+_K$, because $z^1 = N(z)^{-1}\sigma(z)$.

REMARK: Let w be a square-free integer. Consider $\mathcal{O}_K = \mathbb{Z}+\mathbb{Z}w$ as a lattice equipped with the quadratic form $q(z) = N(z)$, and let $\xi \in \mathcal{O}_K$ be a solution of the Pell equation $N(\xi) = 1$. **Then the map $z \mapsto \xi z$ induces an isometry on the lattice $(\mathcal{O}_K, q)$.** In other words, **solutions of the Pelle's equation $N(z) = 1$ are identified with integer points in $SO(q)$.**

Quadratic lattices of rank 2

We define $PSO(\Lambda)$ as the group of isometries of Λ up to ± 1 when rank of Λ is even, and $SO(\Lambda)$ when it is odd.

EXERCISE: Let Λ be an integer lattice of signature (1,1) which represents 0. **Prove that $PSO(\Lambda)$ is trivial.**

Theorem 4: Let (Λ, q) be an integer lattice of signature (1,1) which does not represent 0. **Then $PSO(\Lambda) \cong \mathbb{Z}$.**

Proof. Step 1: Over rational numbers, q can be represented as $q(x,y) = ax^2 - by^2$, where $x, y \in \mathbb{Q}^2 = \Lambda \otimes_{\mathbb{Z}} \mathbb{Q}$. The quadratic form aq can be represented in another rational basis as $q(x_1, y_1) = x_1^2 - wy_1^2$, where $w = ab$. For any solution u of Pell equation $z^2 - t^2 w = 1$, consider the matrix $\begin{pmatrix} z & t \\ tw & z \end{pmatrix}$; it is the matrix of multiplication by $z + t\sqrt{w}$ in $K := \mathbb{Q} + \mathbb{Q}\sqrt{w}$. By multiplicativity of the norm $N(x,y) = x^2 - wy^2$, this matrix induces an isometry of (Λ_1, q), where Λ_1 is the integer lattice in coordinates (x_1, y_1). By construction, Λ is commensurable with Λ_1, **hence $PSO(\Lambda)$ is commensurable with $\mathbb{Z}$.**

Step 2: The Lie group $PSO(\Lambda \otimes_{\mathbb{Z}} \mathbb{R})$ is isomorphic to $\mathbb{R}$ **(prove it)**, and **all finitely generated subgroups of $PSO(\Lambda \otimes_{\mathbb{Z}} \mathbb{R})$ are free abelian.** A free abelian group commensurable with $\mathbb{Z}$ is $\mathbb{Z}$. Therefore, $PSO(\Lambda) \cong \mathbb{Z}$. ■

Rational lines intersecting a rational quadric in irrational points

Let X be a non-degenerate rational quadric in $\mathbb{R}P^2$; in an appropriate coordinate system, X can be given by an equation $P(x,y,z) = ax^2 + by^2 + cy^2 = 0$. We assume that the quadric P is indefinite; in this case X is a circle.

Proposition 1: Let $\mathfrak{I}$ be the set of all irrational points on X obtained by intersecting X with rational lines $\mathbb{R}P^1 \subset \mathbb{R}P^2$. **Then $\mathfrak{I}$ is dense in X.**

Proof. Step 1: Let $(\Lambda_{\mathbb{Q}}, P)$ be the rational quadratic lattice with P as a quadratic form. Reflections centered in rational lines belong to $SO(\Lambda_{\mathbb{Q}})$; an easy geometric argument implies that these reflections act on X with dense orbits. **Therefore, it sufficies to find one irrational point $t \in \mathfrak{I}$; its $SO(\Lambda_{\mathbb{Q}})$-orbit is dense in X.**

Step 2: Reducing to affine coordinates, consider X as a quadric $ax^2+by^2 = 1$ in $\mathbb{R}^2$. If a or b are not full squares, we take a line passing through (0,0) and (0,1) or through (0,0) and (1,0); it intersects with X in a point $(\sqrt{a^{-1}}, 0)$ or in $(0, \sqrt{b^{-1}})$ which is irrational. If both are full squares, we will do another coordinate change, reducing X to the quadric $x^2 + y^2 = 1$, and take a line passing through $(0,0)$ and $(1,1)$. ∎

The null quadric as a limit set

Theorem 3: Let $V_{\mathbb Z}$ be an integer quadratic lattice of signature (a,b), with $a \geqslant 2, b \geqslant 1$, representing g, and $\mathfrak R \subset V_{\mathbb Z}$ the set of all vectors η such that $(\eta,\eta) = g$. **Then the closure of $\mathbb P\mathfrak R \subset \mathbb P V_{\mathbb R}$ contains $\operatorname{Null}(V_{\mathbb R})$.**

Proof: Fix $\eta \in \mathfrak R$. For any $x \in \operatorname{Null}(V_{\mathbb R})$ there exists a 3-dimensional subspace $S \subset V_{\mathbb R}$ of signature $(2,1)$ connecting η and x. We are going to show that x belongs to the closure of the $SO(V_{\mathbb Z})$-orbit of η. Since rational subspaces are dense, it suffices to prove this when S is rational. In this case, **we are going to prove that x belongs to the closure of the $SO(S_{\mathbb Z})$-orbit of η.**

We reduced Theorem 3 to the following statement:
PROPOSITION: Let $V_{\mathbb Z}$ be an integer quadratic lattice of signature $(2,1)$, and $\eta \in \mathbb P V_{\mathbb R}$ a point with positive square. **Then the closure of its $SO(V_{\mathbb Z})$-orbit contains the null quadric $\operatorname{Null}(V_{\mathbb R})$.**

The null quadric as a limit set (2)

We reduced Theorem 3 to the following statement:

PROPOSITION: Let $V_{\mathbb{Z}}$ be an integer quadratic lattice of signature $(2,1)$, and $\eta \in \mathbb{P}V_{\mathbb{R}}$ a point with positive square. **Then the closure of its $SO(V_{\mathbb{Z}})$-orbit contains the null quadric Null($V_{\mathbb{R}}$).**

Proof: Let $S \subset V_{\mathbb{Z}}$ be a sublattice of signature (1,1). If $\mathbb{P}S$ intersects with Null($V_{\mathbb{R}}$) in irrational points, we apply Theorem 4, and obtain that the intersection $\{u_1, u_2\}$ of $\mathbb{P}S_{\mathbb{R}}$ and Null($V_{\mathbb{R}}$) belongs to the closure of $SO(S_{\mathbb{Z}})$-orbit of any $x \in \mathbb{P}S_{\mathbb{R}}$. By Proposition 1, such points $\{u_1, u_2\}$ are dense in Null($V_{\mathbb{R}}$). **Therefore, Null($V_{\mathbb{R}}$) is contained in the closure of any orbit.** ■