K3 surfaces

lecture 15: Lefschetz hyperplane section theorem

Misha Verbitsky

IMPA, sala 236

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Smooth quartic surfaces (reminder)

DEFINITION: **Smooth quartic** is a smooth hypersurface in $\mathbb{C}P^n$, defined by an irreducible homogeneous polynomial of degree 3.

REMARK: By Euler formula, the canonical bundle on $\mathbb{C}P^n$ is $\mathcal{O}(-n-1)$. Adjunction formula applied to a smooth hypersurface $Z \subset \mathbb{C}P^n$ of degree m gives $N^*Z \otimes_{\mathcal{O}_Z} K_Z = K_{\mathbb{C}P^n}|_Z$, where $NZ = \mathcal{O}(m)|_Z$ is the normal bundle. **This gives $K_Z = \mathcal{O}(m-n-1)$.**

COROLLARY: A smooth quartic in $\mathbb{C}P^3$ has trivial canonical bundle. ■

REMARK: In the sequel, **"smooth quartics" will always mean smooth quartic surfaces.**

THEOREM: smooth quartics are diffeomorphic

Proof: Lecture 8. ■

Lefschetz hyperplane section theorem (reminder)

DEFINITION: **Veronese embedding** is the projective embedding $\mathbb{C}P^n \longrightarrow \mathbb{P}(H^0(\mathcal{O}(k)^*)$, defined by the line system $H^0(\mathcal{O}(k))$. In other words, **the Veronese embedding takes**

$$(t_0 : t_1 : ... : t_n) \textbf{ to} (P_0(t_0, ..., t_n) : P_1(t_0, ..., t_n) : ... : ...),$$

where $\{P_i\}$ denotes a basis in homogeneous monomials of degree k.

CLAIM: A smooth quartic is an intersection of a hyperplane and the image of 4-th Veronese embedding of $\mathbb{C}P^3$.

THEOREM: (Lefschetz hyperplane section theorem)
Let $Z \subset \mathbb{C}P^n$ be a smooth projective submanifold of dimension m, and $H \subset \mathbb{C}P^n$ a hyperplane section transversal to Z. Then **for any $i < m-1$, the map of homotopy groups $\pi_i(Z \cap H) \longrightarrow \pi_i(Z)$ is an isomorphism.**

Proof: Later today.

COROLLARY: A smooth quartic Z is a K3 surface.

Proof: Since Z is a hyperplane section of the Veronese manifold, which is isomorphic to $\mathbb{C}P^3$, **Lefschetz theorem gives $\pi_1(Z) = \pi_1(\mathbb{C}P^3) = 0$;** its canonical bundle $K_Z = \mathcal{O}(4-4)|_Z = \mathcal{O}_Z$ vanishes, as shown above. ■

Graded vector spaces and algebras

DEFINITION: A **graded vector space** is a space $V^* = \bigoplus_{i\in\mathbb{Z}} V^i$.

REMARK: If V^* is graded, the endomorphisms space $\operatorname{End}(V^*) = \bigoplus_{i\in\mathbb{Z}} \operatorname{End}^i(V^*)$ is also graded, with $\operatorname{End}^i(V^*) = \bigoplus_{j\in\mathbb{Z}} \operatorname{Hom}(V^j, V^{i+j})$

DEFINITION: A **graded algebra** (or "graded associative algebra") is an associative algebra $A^* = \bigoplus_{i\in\mathbb{Z}} A^i$, with the product compatible with the grading: $A^i \cdot A^j \subset A^{i+j}$.

REMARK: A bilinear map of graded paces which satisfies $A^i \cdot A^j \subset A^{i+j}$ is called **graded**, or **compatible with grading**.

REMARK: The category of graded spaces can be defined as a **category of vector spaces with $U(1)$-action,** with the weight decomposition providing the grading. Then **a graded algebra is an associative algebra in the category of spaces with $U(1)$-action**.

DEFINITION: An operator on a graded vector space is called **even** (**odd**) if it shifts the grading by even (odd) number. The **parity** $\tilde{a}$ of an operator a is 0 if it is even, 1 if it is odd. We say that an operator is **pure** if it is even or odd.

Supercommutator

DEFINITION: A **supercommutator** of pure operators on a graded vector space is defined by a formula $\{a,b\} = ab - (-1)^{\tilde a \tilde b} ba$.

DEFINITION: A graded associative algebra is called **graded commutative** (or "supercommutative") if its supercommutator vanishes.

EXAMPLE: The Grassmann algebra is supercommutative.

DEFINITION: A graded Lie algebra (Lie superalgebra) is a graded vector space $\mathfrak{g}^*$ equipped with a bilinear graded map $\{\cdot,\cdot\}: \ \mathfrak{g}^* \times \mathfrak{g}^* \longrightarrow \mathfrak{g}^*$ which is graded anticommutative: $\{a,b\} = -(-1)^{\tilde a \tilde b}\{b,a\}$ and satisfies **the super Jacobi identity** $\{c,\{a,b\}\} = \{\{c,a\},b\} + (-1)^{\tilde a \tilde c}\{a,\{c,b\}\}$

EXAMPLE: Consider the algebra $\operatorname{End}(A^*)$ of operators on a graded vector space, with supercommutator as above. **Then $\operatorname{End}(A^*), \{\cdot,\cdot\}$ is a graded Lie algebra.**

Lemma 1: Let d be an odd element of a Lie superalgebra, satisfying $\{d,d\} = 0$, and L an even or odd element. **Then $\{\{L,d\},d\} = 0$.**

Proof: $0 = \{L,\{d,d\}\} = \{\{L,d\},d\} + (-1)^{\tilde L}\{d,\{L,d\}\} = 2\{\{L,d\},d\}$. ■

The twisted differential d^c

DEFINITION: The **twisted differential** is defined as $d^c := IdI^{-1}$.

CLAIM: Let (M, I) be a complex manifold. **Then $\partial := \frac{d+\sqrt{-1}\,d^c}{2}$, $\overline{\partial} := \frac{d-\sqrt{-1}\,d^c}{2}$ are the Hodge components of d**, $\partial = d^{1,0}$, $\overline{\partial} = d^{0,1}$.

Proof: The Hodge components of d are expressed as $d^{1,0} = \frac{d+\sqrt{-1}\,d^c}{2}$, $d^{0,1} = \frac{d-\sqrt{-1}\,d^c}{2}$. Indeed, $I(\frac{d+\sqrt{-1}\,d^c}{2})I^{-1} = \sqrt{-1}\frac{d+\sqrt{-1}\,d^c}{2}$, hence $\frac{d+\sqrt{-1}\,d^c}{2}$ **has Hodge type (1,0);** the same argument works for $\overline{\partial}$. ■

CLAIM: Let W be **the Weil operator**, $W\Big|_{\Lambda^{p,q}(M)} = \sqrt{-1}\,(p-q)$. On any complex manifold, one has $d^c = [W, d]$.

Proof: Clearly, $[W, d^{1,0}] = \sqrt{-1}\,d^{1,0}$ and $[W, d^{0,1}] = -\sqrt{-1}\,d^{0,1}$. Then $[W, d] = \sqrt{-1}\,d^{1,0} - \sqrt{-1}\,d^{0,1} = IdI^{-1}$. ■

COROLLARY: $\{d, d^c\} = \{d, \{d, W\}\} = 0$

Proof: Implied by Lemma 1. ■

Plurilaplacian

THEOREM: Let (M,I) be a complex manifold. **Then 1.** $\partial^2 = 0$.
2. $\overline{\partial}^2 = 0$.
3. $dd^c = -d^c d$
4. $dd^c = 2\sqrt{-1}\,\partial\overline{\partial}$.

Proof: The first is vanishing of (2,0)-part of d^2, and the second is vanishing of its (0,2)-part. Now, $\{d, d^c\} = -\{d,\{d,W\}\} = 0$ (Lemma 1), this gives $dd^c = -d^c d$. Finally, $2\sqrt{-1}\partial\overline{\partial} = \frac{1}{2}(d+\sqrt{-1}d^c)(d-\sqrt{-1}d^c) = \frac{1}{2}(dd^c - d^c d) = dd^c$. ■

DEFINITION: The operator dd^c is called **the pluri-Laplacian**.

REMARK: The pluri-Laplacian **takes real functions to real (1,1)-forms on M.**

EXERCISE: Prove that **on a Riemannian surface (M,I,ω), one has $dd^c(f) = \Delta(f)\omega$.**

DEFINITION: **The Hodge $U(1)$-action** on differential forms on a complex manifold defined by $\rho(t)(\eta) = e^{tW}(\eta)$. On (p,q)-forms, it acts as a scalar $\rho(t)\Big|_{\Lambda^{p,q}(M)} = e^{(p-q)\sqrt{-1}}\operatorname{Id}$; the (p,p)-forms are clearly invariant.

Positive (1,1)-forms

CLAIM: Consider a real (1,1)-form $\eta \in \Lambda^{1,1}(M) \cap \Lambda^2(M,\mathbb{R})$. **Then the bilinear form $g_\eta(x,y) := \eta(x,Iy)$ is symmetric.**

Proof: Clearly, $0 = W(\eta)(x,y) = \eta(W(x),y)+\eta(x,W(y)) = \eta(Ix,y)+\eta(x,Iy)$. This gives $\eta(x,Iy) = -\eta(Ix,y) = \eta(y,Ix)$. ■

REMARK: This construction is reversible, and **defines a bijection between $U(1)$-invariant symmetric forms $g \in \operatorname{Sym}^2(T^*M)$ and sections of $\Lambda^{1,1}(M)\cap \Lambda^2(M,\mathbb{R})$.**

DEFINITION: A real (1,1)-form η is called **positive** if $\eta(x,Ix) \geqslant 0$ for any $x \in TM$. By this convention, **0 is a positive (1,1)-form ("French positive")**

DEFINITION: A (1,1)-form is called **Hermitian** if it is positive and non-degenerate, that is, when $\eta(x,Ix) > 0$ for any $x \in TM\backslash 0$.

REMARK: The above construction **gives a bijective correspondence between the Hermitian (1,1)-forms and $U(1)$-invariant Riemannian metric tensors on M.**

EXAMPLE: For any $(1,0)$-form ξ, **the form $\sqrt{-1}\,\xi\wedge\bar\xi$ is positive (prove this).**

Pluri-harmonic functions

DEFINITION: A function f on a complex manifold is called **pluri-harmonic** if $dd^2 f = 0$.

REMARK: A function f is called **holomorphic** if $\overline{\partial} f = 0$, and **antiholomorphic** if $\partial f = 0$. Since $dd^c = 2\sqrt{-1}\,\partial\overline{\partial} = -2\sqrt{-1}\,\overline{\partial}\partial$, **any holomorphic and any antiholomorphic function is pluri-harmonic.**

THEOREM: Any pluriharmonic function **is locally expressed as a sum of holomorphic and antiholomorphic function.**

Proof: Let f be a pluriharmonic function on a ball, and $\alpha = \partial f$. Since $\overline{\partial}(\alpha) = 0$, this form is holomorphic; since $\partial^2 = 0$, it is also closed. Poincaré lemma applied to holomorphic functions implies that $\alpha = du$, where u is holomorphic. Then $d(f-u)$ is a (0,1)-form, hence $v := f-u$ is antiholomorphic. **We obtain that $f = u + v$, where u is holomorphic, and v is antiholomorphic.** ■

In our proof, we use the following lemma.

LEMMA: Let $B \subset \mathbb{C}^n$ be an open ball, and η a closed holomorphic $(d, 0)$-form **Then $\eta = d\alpha$, where α is a holomorphic $(d-1, 0)$-form.**

Proof: It follows from the Poincaré lemma directly. ■

Morse functions

DEFINITION: Let f be a smooth function on a manifold M, and $x \in M$ its critical point. Choose a coordinate system $x_1, ..., x_n$ in a neighbourhood of x. **The hessian** of a function f in x is a symmetric matrix $\operatorname{Hess}(f) := \sum_i \frac{d^2 f}{dx_i dx_j} \cdot dx_i \otimes dx_j \in \operatorname{Sym}^2 M$.

CLAIM: The form $\operatorname{Hess}(f)$ **is coordinate-independent, and defines a symmetric 2-form on $T_x M$.**

Proof: Do this as an exercise. ■

DEFINITION: A smooth function $f : M \longrightarrow \mathbb{R}$ is called **Morse** if it is proper (that is, the preimage of a closed interval is compact), its critical points are isolated, and for each of these critical point, **the form** $\operatorname{Hess}(f)$ **is non-degenerate.**

CLAIM: **Every manifold admits a Morse function.** Moreover, **the set of Morse functions is dense and open in the space of all proper smooth functions taken with C^2 or C^∞-topology.**

Proof: *Milnor, "Morse theory."* ■

The Hessian and torsion-free connections

DEFINITION: Let $\text{Alt}: \Lambda^1 M \otimes \Lambda^1 M \longrightarrow \Lambda^2 M$ denote the antisymmetrization map $x \otimes y \mapsto x \wedge y$. A connection $\nabla: \Lambda^1 M \longrightarrow \Lambda^1 M \otimes \Lambda^1 M$ is called **torsion-free**, if $\text{Alt}(\nabla\theta) = d\theta$ for any 1-form θ on M.

CLAIM: Let (M, ∇) be a manifold with a torsion-free connection, and φ a function. **Then the 2-form $\text{Hess}(\varphi) := \nabla(d\varphi) \in \Lambda^1 M \otimes \Lambda^1 M$ is symmetric.**
Proof: Clear. ■

REMARK: This claim immediately implies that **the form $\nabla(df)$ is symmetric for any function f on M.**

CLAIM: Let f be a smooth function on a manifold, and x its critical point. **Then $\nabla(df) = \text{Hess}(f)$ on $T_x M$.**
Proof. Step 1: A difference of two connections $\nabla - \nabla_1$ is a 1-form $A \in \Lambda^1(M) \otimes \text{End}(\Lambda^1 M)$. Since $df|_x = 0$, we have $\nabla + A(df)|_x = \nabla(df) + A(df)|_x = \nabla(df)|_x$. Therefore, **$\nabla(df)|_x$ is independent from ∇.**

Step 2: Fix the coordinate system $\{x_i\}$ in a neighbourhood of x, and take the connection $\nabla_1(\theta) := \sum_i \frac{d\theta}{dx_i} \otimes dx_i$, where $\frac{d\theta}{dx_i}$ denotes the derivative of all coefficients. By definition, $\nabla_1(df)|_x = \sum_i \frac{d^2 f}{dx_i dx_j} \cdot dx_i \otimes dx_j|_x = \text{Hess}(f)$. **By Step 1, this quantity is equal to $\nabla(df)|_x$.** ■

Levi-Civita connection

DEFINITION: Let (M,g) be a Riemannian manifold. **A Levi-Civita connection** is a connection on TM which is torsion-free and orthogonal, that is, satisfies $\nabla(g)=0$.

THEOREM: The Levi-Civita connection **exists and is unique on any Riemannian manifold.**

Proof: http://verbit.ru/IMPA/HK-2023/slides-hk-2023-02.pdf ■

THEOREM: Let (M,I,g) be a Kähler manifold. **Then the Levi-Civita connection satisfies $\nabla(I)=0$.** Conversely, if the Levi-Civita connection on an almost complex Hermitian manifold (M,I,g) satisfies $\nabla(I)=0$, **this manifold is Kähler.**

Proof: http://verbit.ru/MATH/KAHLER-2020/slides-Kahler-2020-10.pdf ■

Torsion-free connections preserving the complex structure

EXERCISE: Let (M, I) be an almost complex manifold, and ∇ a torsion-free connection on TM preserving I, that is, satisfying $\nabla(I) = 0$. **Prove that I is integrable.**

EXERCISE: Let (M, I) be a complex manifold. **Prove that M admits a torsion-free connection ∇ preserving I.**

REMARK: Locally in M, **this statement is trivial,** because the same trivial connection written in coordinates as above,

$$\nabla_1(\theta) := \sum_i \frac{d\theta}{dx_i} \otimes dx_i$$

preserves the complex structure.

The pluri-Laplacian and the Hessian

DEFINITION: Let (M,I) be a complex manifold, and $d^c := I^{-1}dI : \Lambda^i(M) \to \Lambda^{i+1}(M)$ the twisted differential. Let $f \in C^\infty M$ be a real function. If $dd^c f(X, IX) \geqslant 0$ for all X, the function f is called **plurisubharmonic** (psh).

REMARK: Let ∇ be a flat, torsion-free connection on M, and $\operatorname{Alt} : T^*M \otimes T^*M \longrightarrow \Lambda^2(M)$ be the antisymmetrizer. **Then**

$$dd^c f = d(Id(f)) = \operatorname{Alt}(\nabla I(df)) = \operatorname{Alt}(\operatorname{Id} \otimes I(\operatorname{Hess}(f)),$$

where $\operatorname{Hess}(f) = \nabla(df) \in T^*M \otimes T^*M$.

COROLLARY 1: Consider a complex manifold equipped with a torsion-free connection preserving a complex structure I. **Then**

$$dd^c f(x, Ix) = \frac{1}{2}(\operatorname{Hess}(f)(x,x) + \operatorname{Hess}(f)(Ix, Ix)).$$

■

The Morse index of a plurisubharmonic function

DEFINITION: Let x be a Morse critical point of $f \in C^\infty M$, and (u,v) is signature of its Hessian. The **Morse index** of f in x is v.

PROPOSITION: Let f be a plurisubharmonic Morse function on a complex manifold M, $\dim_{\mathbb{C}} M = n$, and $m \in M$ its Morse point. **Then its Morse index is $\leqslant n$.**

Proof. Step 1: The Morse index is the dimension of the biggest subspace $W \subset T_m M$ such that $\operatorname{Hess}(f)$ is negative definite on W. Assume that $\dim W > n$. **Then $W_1 := W \cap I(W)$ is positive-dimensional.**

Step 2: For any non-zero $x \in W_1$, we have $I(x) \in W$, hence $\operatorname{Hess}(f)(Ix, Ix) < 0$. Then $dd^c f(x, Ix) = \frac{1}{2}(\operatorname{Hess}(f)(x,x) + \operatorname{Hess}(f)(Ix, Ix)) < 0$, giving a contradiction. ■

Stable manifold of a critical point

DEFINITION: Let f be a Morse function on a smooth manifold M, and $\operatorname{grad} f$ its gradient vector field. **The stable manifold** of a critical point m is all points $z \in M$ such that $\lim\limits_{t \longrightarrow \infty} e^{t \operatorname{grad} f}(z) = m$.

PROPOSITION: Let Z_m be a stable manifold of a critical point $m \in M$ of index p. **Then Z_m is a smooth, p-dimensional submanifold in M**

Proof: "Morse lemma" (see any textbook on Morse theory, such as Milnor's) claims that there is a coordinate system $x_1, .., x_n$ in a neighbourhood of m such that $f = \sum_{i=1}^{n-p} x_i^2 - \sum_{i=n-p+1}^{n} x_i^2$. **The map $z \to e^{t \operatorname{grad} f}(z)$ as $t \to \infty$ smoothly retracts the stable manifold to the set $(0, 0, ..., 0, x_{n-p+1}, ..., x_n)$ which is smooth.** ■

REMARK: Pushing this argument further, **it is possible to construct the cell decomposition of M, with the p-dimensional cells in bijective correspondence with Morse points of index p**; this argument lies in the foundations of **Morse theory.**

Lefschetz hyperplane section theorem

THEOREM: (Lefschetz hyperplane section theorem)
Let $Z \subset \mathbb{C}P^n$ be a smooth projective submanifold of dimension m, and $H \subset \mathbb{C}P^n$ a hyperplane section transversal to Z. Then **for any $i < m-1$, the map of homotopy groups $\pi_i(Z \cap H) \longrightarrow \pi_i(Z)$ is an isomorphism.**

Proof. Step 1: Consider the function $f_1 := |z^2| = \sum_i |x_i|^2 + |y_i^2|$ on $\mathbb{C}P^n \backslash H = \mathbb{C}^n$. Since $dd^c f_1 = 2\sum_i dx_i \wedge dy_i$, this function is strictly plurisubharmonic. The strictly plurisubharmonic functions are open in C^2-topology, and Morse function are dense in C^2-topology, hence **there exists a small deformation f of f_1 which is strictly plurisubharmonic and Morse on $Z \cap \mathbb{C}P^n \backslash H$.**

Step 2: Let $\{V_i \subset Z\backslash(H \cap Z)\}$ be all stable sets of all critical points of f on Z. Then **the intersection $Z \cap H$ is a deformational retract of $Z_0 := Z \backslash \bigcup_i V_i$**: the map $z \longrightarrow \lim_{t\to\infty} e^{t \operatorname{grad} f}(z)$ retracts Z_0 to $Z \cap H$. Indeed, the limit of $e^{t \operatorname{grad} f}(z)$ on Z is either $Z \cap H$, or a critical point of f, and in the latter case, z belongs to its stable set.

Lefschetz hyperplane section theorem (2)

THEOREM: (Lefschetz hyperplane section theorem)
Let $Z \subset \mathbb{C}P^n$ be a smooth projective submanifold of dimension m, and $H \subset \mathbb{C}P^n$ a hyperplane section transversal to Z. Then **for any $i < m-1$, the map of homotopy groups $\pi_i(Z \cap H) \longrightarrow \pi_i(Z)$ is an isomorphism.**

Proof. Step 1 (abbreviated): Consider the function $f_1 := |z^2| = \sum_i |x_i|^2 + |y_i^2|$ on $\mathbb{C}P^n \backslash H = \mathbb{C}^n$. A small perturbation makes this function Morse.

Step 2 (abbreviated): Let $\{V_i \subset Z \backslash (H \cap Z)\}$ be all stable sets of all critical points of f on Z. Then **the intersection $Z \cap H$ is a deformational retract of $Z_0 := Z \backslash \bigcup_i V_i$.**

Step 3: Since f is strictly plurisubharmonic, the index of its critical points is $\leqslant \dim_{\mathbb{C}} Z$. By the previous step, the space $Z_0 = Z \backslash \bigcup_i V_i$ is homotopy equivalent to $H \cap Z$. Therefore, **Lefschetz hyperplane section theorem is implied by the following topological lemma.**

LEMMA: Let Z be a smooth manifold, $V_i \subset Z$ a collection of smooth submanifolds of Z, $\operatorname{codim} \dim V_i \geqslant m$, and $Z_0 := Z \backslash \bigcup_j V_j$. Then the natural embedding **$Z_0 \hookrightarrow Z$ induces an isomorphism of homotopy groups** $\pi_i(Z_0) \cong \pi_i(Z)$ for all $i < m-1$, and is surjective for $i = m-1$.

How $\pi_i(M)$ is affected by removing a submanifold $Z \subset M$

LEMMA: Let Z be a smooth manifold, $V_i \subset Z$ a collection of smooth submanifolds of Z, $\operatorname{codim}\dim V_i \geqslant m$, and $Z_0 := Z \backslash \bigcup_j V_j$. Then the natural embedding $Z_0 \hookrightarrow Z$ **induces an isomorphism of homotopy groups** $\pi_i(Z_0) \cong \pi_i(Z)$ for all $i < m-1$, and is surjective for $i = m-1$.

Proof. Step 1: To show that $\pi_i(Z_0) \xrightarrow{\rho} \pi_i(Z)$ is surjective, take any element $\pi_i(Z)$, represent it by an immersion of a sphere $S^i \longrightarrow Z$, and deform this sphere so that it becomes transversal to V_j, for all j. **If $\operatorname{codim} V_j > i$, transversality implies that $S^i \cap V_j = 0$, hence the image of ρ belongs to Z_0.** This implies that the natural map $\pi_i(Z_0) \cong \pi_i(Z)$ is surjective.

Step 2: Let $\rho_0 := S^i \longrightarrow Z_0$ be a sphere map which is homotopic to zero in Z. **This homotopy can be expressed as a map from an $i+1$-dimensional ball $B^{i+1} \xrightarrow{\rho} Z$, such that $\rho\big|_{\partial B^{i+1}} = \rho_0$.** By transversality theorem, the map ρ can be chosen smooth and transversal to all V_j. If, in addition, $i+1 < \operatorname{codim} V_i$, the image of ρ does not intersect $\bigcup_j V_j$, which implies that ∂B^{i+1} is homotopic to zero in Z_0. ■