K3 surfaces

lecture 16: C-symplectic Moser lemma and the local Torelli theorem

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October 23, 2024, 17:00

C-symplectic structures (reminder)

DEFINITION: Let M be a smooth $4n$-dimensional manifold. A closed complex-valued form Ω on M is called **C-symplectic** if $\Omega^{n+1} = 0$ and $\Omega^n \wedge \overline{\Omega}^n$ is a non-degenerate volume form.

THEOREM: Let $\Omega \in \Lambda^2(M, \mathbb{C})$ be a C-symplectic form, and $T^{0,1}_\Omega(M)$ be equal to $\ker \Omega$, where

$$\ker \Omega := \{v \in TM \otimes \mathbb{C} \ \mid \ \Omega \lrcorner v = 0\}.$$

Then $T^{0,1}_\Omega(M) \oplus \overline{T^{0,1}_\Omega(M)} = TM \otimes_{\mathbb{R}} \mathbb{C}$, hence **the sub-bundle $T^{0,1}_\Omega(M)$ defines an almost complex structure I_Ω on M.** If, in addition, **Ω is closed, I_Ω is integrable, and Ω is holomorphically symplectic on (M, I_Ω).**

Proof: Rank of Ω is $2n$ because $\Omega^{n+1} = 0$ and $\operatorname{Re} \Omega$ is non-degenerate. Then $\ker \Omega \oplus \overline{\ker \Omega} = T_{\mathbb{C}} M$. The relation $[T^{0,1}_\Omega(M), T^{0,1}_\Omega(M)] \subset T^{0,1}_\Omega(M)$ follows from Theorem 1 on the next slide. ∎

Closed forms and integrable distributions (reminder)

Theorem 1: Let Ω be a p-form on a smooth manifold, and $B = \ker \Omega$. **Then $[B, B] \subset B$,** that is, **the distribution B is integrable.**

Proof. Step 1: Cartan's formula implies that $\operatorname{Lie}_X(\Omega) = d(i_X(\Omega)) + i_x(d\Omega) = 0$, **hence $\operatorname{Lie}_X(\Omega) = 0$ for any $X \in B$.**

Step 2: Let $X, X_1 \in B$, and $X_2, ..., X_p$ any vector fields. Cartan formula gives

$$\operatorname{Lie}_X(\Omega)(X_1, ..., X_p) = \operatorname{Lie}_X(\Omega(X_1, ..., X_p)) - \sum_{i=1}^{p} \Omega(X_1, ..., [X, X_i], ...X_p).$$

All terms of this sum, except $\Omega([X, X_1], X_2, ..., X_p)$, vanish, because $X_1 \in B$ and $\operatorname{Lie}_X(\Omega) = 0$. **This gives $\Omega([X, X_1], X_2, ..., X_p) = 0$ for all $X_2, ..., X_p$.** Therefore, $[X, X_1] \in B$. ■

COROLLARY: Let (M, I) be an almost complex manifold admitting a holomorphic symplectic form. **Then I is integrable.**

Proof: Indeed, $\ker \Omega = T^{0,1}(M)$. ■

Holomorphically symplectic Teichmüller space (reminder)

DEFINITION: Let CSymp be the space of all C-symplectic forms on a manifold M, equipped with the C^∞-topology, and Diff_0 the connected component of the group of diffeomorphisms. The **holomorphically symplectic Teichmüller space** CTeich is the quotient $\frac{\operatorname{CSymp}}{\operatorname{Diff}_0}$.

REMARK: Recall that **the mapping class group** of a manifold M is the group $\Gamma := \frac{\operatorname{Diff}}{\operatorname{Diff}_0}$ of connected components of $\operatorname{Diff}(M)$.

REMARK: The quotient $\operatorname{CTeich}/\Gamma$ **is identified with the set of all holomorphically symplectic structures on M up to isomorphism.**

Period map for holomorphically symplectic manifolds (reminder)

DEFINITION: Let (M, I, Ω) be a holomorphically symplectic manifold, and CSymp the space of all C-symplectic forms. The quotient CTeich $:= \frac{\text{CSymp}}{\text{Diff}_0}$ is called **the holomorphically symplectic Teichmüller space**, and the map CTeich $\longrightarrow H^2(M, \mathbb{C})$ taking (M, I, Ω) to the cohomology class $[\Omega] \in H^2(M, \mathbb{C})$ **the holomorphically symplectic period map**.

THEOREM: (Local Torelli theorem, due to Bogomolov) Let (M, I, Ω) be a complex, Kähler, holomorphically symplectic surface with $H^{0,1}(M) = 0$, that is, a K3 surface. Consider the period map

$$\text{Per}: \ \text{CTeich} \longrightarrow H^2(M, \mathbb{C})$$

taking (M, I, Ω) to the cohomology class $[\Omega] \in H^2(M, \mathbb{C})$. **Then** Per **is a local diffeomorpism** of CTeich to the **period space**

$$Q := \left\{ v \in H^2(M, \mathbb{C}) \quad | \quad \int_M v \wedge v = 0, \int_M v \wedge \overline{v} > 0 \right\}.$$

Proof: Next lecture

A caution: CTeich **is smooth, but non-Hausdorff.** The non-Hausdorff points are well understood and correspond to the partition of the "positive cone" $\{v \in H^{1,1}_I(M, \mathbb{R}) \ \ | \ \ \int_M v \wedge v > 0\}$ onto "Kähler chambers" (to be explained later).

C-symplectic structures on surfaces

CLAIM: In real dimension 4, C-symplectic structure **is determined by a pair $\omega_1 = \operatorname{Re}\Omega, \omega_2 = \operatorname{Im}\Omega$ of symplectic forms which satisfy $\omega_1^2 = \omega_2^2$ and $\omega_1 \wedge \omega_2 = 0$.**

Proof: Let Ω be a C-symplectic form, $\omega_1 = \operatorname{Re}\Omega$ and $\omega_2 = \operatorname{Im}\Omega$. Then $\Omega \wedge \Omega = \omega_1^2 + \omega_2^2 + 2\sqrt{-1}\,\omega_1 \wedge \omega_2 = 0$, hence $\omega_1^2 = \omega_2^2$ and $\omega_1 \wedge \omega_2 = 0$. The form $\Omega \wedge \overline{\Omega} = \omega_1^2 + \omega_2^2$ is non-degenerate, hence $\omega_1^2 = \omega_2^2$ is non-degenerate.

Conversely, if $\omega_1^2 = \omega_2^2$ and $\omega_1 \wedge \omega_2 = 0$, we have $\Omega \wedge \Omega = 0$, and $\Omega \wedge \overline{\Omega} = \omega_1^2 + \omega_2^2$ is non-degenerate if ω_i is non-degenerate. ■

REMARK: For K3 surface, the local Torelli theorem is equivalent to the following statement: the period map $\operatorname{Per}: \text{CTeich} \longrightarrow H^2(M, \mathbb{C})$ which takes ω_1, ω_2 to their cohomology classes which satisfy $[\omega_1]^2 = [\omega_2]^2$ and $[\omega_1] \wedge [\omega_2] = 0$ **is locally a diffeomorphism.**

Compare this with Moser theorem (next slide).

Moser theorem (reminder)

DEFINITION: Let $\operatorname{Teich}_s := \operatorname{Symp} / \operatorname{Diff}_0$ be the Teichmüller space of symplectic structures on M. Define **the period map** $\operatorname{Per}: \operatorname{Teich}_s \longrightarrow H^2(M, \mathbb{R})$ mapping a symplectic structure to its cohomology class.

THEOREM: (Moser, 1965)
The **Teichmüler space** Teich_s **is a manifold** (possibly, non-Hausdorff), and the **period map** $\operatorname{Per}: \operatorname{Teich}_s \longrightarrow H^2(M, \mathbb{R})$ **is locally a diffeomorphism.**

The proof is based on another theorem of Moser.

THEOREM: (Moser isotopy lemma)
Let ω_t, $t \in S$ be a smooth family of symplectic structures, parametrized by a connected manifold S. Assume that the cohomology class $[\omega_t] \in H^2(M)$ is constant in t. **Then all ω_t are isotopic.**

Proof of Moser theorem: The period map $P: \operatorname{Symp} \longrightarrow H^2(M, \mathbb{R})$ is a smooth submersion. By Moser isotopy lemma, the conneced components of the fibers of P are orbits of $\operatorname{Diff}_0(M)$. Therefore, P is locally a diffeomorphism. ■

Moser isotopy lemma

THEOREM: (Moser's isotopy lemma)
Let M be a compact symplectic manifold, and ω_t, $t \in [0,1]$ a smooth deformation of a symplectic form. Assume that the cohomology class $[\omega_t] \in H^2(M)$ is constant in t. **Then there exists a diffeomorphism flow $\Psi_t \in \operatorname{Diff}(M)$ mapping ω_t to ω_0, for all t.**

Proof. Step 1: Since all ω_t are cohomologous, the form $\frac{d\omega_t}{dt}$ is exact. Then $\frac{d\omega_t}{dt} = d\eta_t$, where $\eta_t \in \Lambda^1(M)$. **Using Hodge theory, this form can be chosen smoothly in t.**

Step 2: Let v_t be the tangent vector field to Ψ_t, with $v_t := \Psi_t^{-1}\frac{d\Psi_t}{dt}$. Assume that $\Psi_0 = \operatorname{Id}$. The equation $\Psi_t^*\omega_t = \omega_0$ (for all $t \in [0,1]$) is equivalent to $\frac{d}{dt}\Psi_t^*\omega_t = 0$, equivalently, $\frac{d\Psi_t}{dt}\omega_t = -\Psi_t\frac{d\omega_t}{dt}$, which is the same as

$$\operatorname{Lie}_{v_t}\omega_t = -\frac{d\omega_t}{dt}. \quad (*)$$

By Cartan's formula, $\operatorname{Lie}_{v_t}\omega_t = d(i_{v_t}(\omega_t))$. **Then (*) is equivalent to $d(i_{v_t}\omega_t) = -d\eta_t$.**

Step 3: Since ω_t is non-degenerate, there exists a unique $v_t \in TM$ such that $i_{v_t}\omega_t = -\eta_t$. Integrating the time-dependent vector field v_t to a flow of diffeomorphisms, **we obtain Ψ_t satisfying $\Psi_t^*\omega_t = \omega_0$.** ■

C-symplectic Moser lemma

THEOREM: (C-symplectic Moser lemma)
Let (M, I_t, Ω_t), $t \in [0,1]$ be a family of C-symplectic forms on a compact manifold. Assume that the cohomology class $[\Omega_t] \in H^2(M, \mathbb{C})$ is constant, and $H^{0,1}(M, I_t) = 0$, where $H^{0,1}(M, I_t) = H^1(M, \mathcal{O}_{(M,I_t)})$ is cohomology of the sheaf of holomorphic functions. Then **there exists a smooth family of diffeomorphisms $V_t \in \operatorname{Diff}_0(M)$, such that $V_t^* \Omega_0 = \Omega_t$.**

Proof. Step 1: If we find a vector field X_t such that $\operatorname{Lie}_{X_t} \Omega_t = \frac{d}{dt}\Omega_t$, we have (like in the proof of Moser's lemma)

$$V_{t_1}^* \Omega_0 = \int_0^{t_1} \operatorname{Lie}_{X_t} \Omega_t dt = \int_0^{t_1} \frac{d\Omega_t}{dt} dt = \Omega_{t_1}$$

where V_t is a diffeomorphism flow such that $\frac{dV_t}{dt} = X_t$. **It remains to find the family $X_t \in T_{\mathbb{R}} M$.**

Step 2: The contraction map $T_{\mathbb{R}} M \longrightarrow \Lambda^{1,0}(M)$ taking x to $i_x \Omega$ **is surjective** (an exercise in linear algebra).

Step 3: Since $\frac{d}{dt}\Omega_t$ is exact, one has $\frac{d}{dt}\Omega_t = d\alpha_t$. If α_t has Hodge type (1,0), we could obtain it as $\Omega_t \lrcorner X_t$ (Step 2), which gives $\frac{d}{dt}\Omega_t = d\alpha_t = d(\Omega_t \lrcorner X_t) = \operatorname{Lie}_{X_t} \Omega_t$. **It remains to find $\alpha_t \in \Lambda^{1,0}(M, I_t)$ such that $\frac{d}{dt}\Omega_t = d\alpha_t$.**

C-symplectic Moser's lemma (2)

It remains to find $X_t \in T_{\mathbb R}M$ such that $\operatorname{Lie}_{X_t}\Omega_t = \frac{d}{dt}\Omega_t$.

Step 2: The contraction map $\Lambda^{2,0}M \otimes_{\mathbb R} T_{\mathbb R}M \longrightarrow \Lambda^{1,0}(M)$ **is surjective**.

Step 3: Since $\frac{d}{dt}\Omega_t$ is exact, one has $\frac{d}{dt}\Omega_t = d\alpha_t$. If α_t has Hodge type (1,0), we could obtain it as $\Omega_t \lrcorner X_t$ (Step 2), which gives $\frac{d}{dt}\Omega_t = d\alpha_t = d(\Omega_t \lrcorner X_t) = \operatorname{Lie}_{X_t}\Omega_t$. **It remains to find $\alpha_t \in \Lambda^{1,0}(M, I_t)$ such that $\frac{d}{dt}\Omega_t = d\alpha_t$.**

Step 4: Let $\Omega_t' := \frac{d}{dt}\Omega_t$ and $\dim_{\mathbb C} M = 2n$. Differentiating $\Omega_t^{n+1} = 0$ in t, we obtain $\Omega_t' \wedge \Omega_t^n = 0$. Since $\Phi := \Omega_t^n$ is a holomorphic volume form, the multiplication map $\Lambda^{0,2}(M) \xrightarrow{\wedge\Phi} \Lambda^{2n,2}(M)$ is an isomorphism of vector bundles. **Then $\Omega_t' \wedge \Omega_t^n = 0$ is equivalent to $\Omega_t' \in \Lambda^{1,1}(M, I_{\Omega_t}) + \Lambda^{2,0}(M, I_{\Omega_t})$.**

Step 5: Using Step 3 and Step 4, we obtain that holomorphic Moser's lemma **is implied by the following statement.**

LEMMA: Let M be a complex manifold which satisfies $H^{0,1}(M) = 0$, and $\eta \in \Lambda^{1,1}(M) + \Lambda^{2,0}(M)$ an exact form. **Then $\eta = d\alpha$, for some $\alpha \in \Lambda^{1,0}(M)$.**

C-symplectic Moser's lemma (3)

LEMMA: Let M be a complex manifold which satisfies $H^{0,1}(M) = 0$, and $\eta \in \Lambda^{1,1}(M) + \Lambda^{2,0}(M)$ an exact form. **Then $\eta = d\alpha$, for some $\alpha \in \Lambda^{1,0}(M)$.**

Proof. Step 1: Let $\eta = d\beta$, where $\beta = \beta^{1,0} + \beta^{0,1}$. Since $\eta \in \Lambda^{1,1}(M) + \Lambda^{2,0}(M)$, we have $\overline{\partial}(\beta^{0,1}) = 0$. The first cohomology of the complex $(\Lambda^{0,*}(M), \overline{\partial})$ vanish, because $H^{0,1}(M) = 0$, **hence $\beta^{0,1} = \overline{\partial}\psi$, for some $\psi \in C^\infty M$.**

Step 2: This gives $\eta = d(\beta - d\psi)$, hence $\alpha := \beta - d\psi = \beta^{1,0} + \beta^{0,1} - \partial\psi - \beta^{0,1}$ **is a (1,0)-form which satisfies $\eta = d\alpha$.** ■

The following result will follow after we prove that Per is a smooth submersion.

COROLLARY: Let CSymp be the space of all C-symplectic structures with C^∞-topology. Denote by Teich_C the corresponding Teichmüller space, $\text{Teich}_C := \frac{\text{CSymp}}{\text{Diff}_0(M)}$. Define **the period map** $\text{Per}: \text{Teich}_C \longrightarrow H^2(M, \mathbb{C})$ mapping Ω to its cohomology class. **Then Per is locally a homeomorphism to its image.**

Proof: All fibers of Per are 0-dimensional, hence Per is a covering, but all coverings of a ball are diffeomorphic on each connected component. ■