K3 surfaces

lecture 17, local Torelli theorem: local surjectivity of the period map

Misha Verbitsky

IMPA, sala 236

October 28, 2024, 17:00

C-symplectic structures (reminder)

DEFINITION: Let M be a smooth $4n$-dimensional manifold. A closed complex-valued form Ω on M is called **C-symplectic** if $\Omega^{n+1} = 0$ and $\Omega^n \wedge \overline{\Omega}^n$ is a non-degenerate volume form.

THEOREM: Let $\Omega \in \Lambda^2(M, \mathbb{C})$ be a C-symplectic form, and $T^{0,1}_\Omega(M)$ be equal to $\ker \Omega$, where

$$\ker \Omega := \{v \in TM \otimes \mathbb{C} \ \mid \ \Omega \lrcorner v = 0\}.$$

Then $T^{0,1}_\Omega(M) \oplus \overline{T^{0,1}_\Omega(M)} = TM \otimes_{\mathbb{R}} \mathbb{C}$, hence **the sub-bundle $T^{0,1}_\Omega(M)$ defines an almost complex structure I_Ω on M.** If, in addition, **Ω is closed, I_Ω is integrable, and Ω is holomorphically symplectic on (M, I_Ω).**

DEFINITION: Let CSymp be the space of all C-symplectic forms on a manifold M, equipped with the C^∞-topology, and Diff_0 the connected component of the group of diffeomorphisms. The **holomorphically symplectic Teichmüller space** CTeich is the quotient $\frac{\text{CSymp}}{\operatorname{Diff}_0}$.

Period map for holomorphically symplectic manifolds (reminder)

DEFINITION: Let (M, I, Ω) be a holomorphically symplectic manifold, and CSymp the space of all C-symplectic forms. The quotient $\mathsf{CTeich} := \frac{\mathsf{CSymp}}{\mathsf{Diff}_0}$ is called **the holomorphically symplectic Teichmüller space**, and the map $\mathsf{CTeich} \longrightarrow H^2(M, \mathbb{C})$ taking (M, I, Ω) to the cohomology class $[\Omega] \in H^2(M, \mathbb{C})$ **the holomorphically symplectic period map**.

THEOREM: (Local Torelli theorem, due to Bogomolov) Let (M, I, Ω) be a complex, Kähler, holomorphically symplectic surface with $H^{0,1}(M) = 0$, that is, a K3 surface. Consider the period map

$$\mathsf{Per}: \ \mathsf{CTeich} \longrightarrow H^2(M, \mathbb{C})$$

taking (M, I, Ω) to the cohomology class $[\Omega] \in H^2(M, \mathbb{C})$. **Then Per is a local diffeomorpism** of CTeich to the **period space**

$$Q := \left\{ v \in H^2(M, \mathbb{C}) \quad | \quad \int_M v \wedge v = 0, \int_M v \wedge \overline{v} > 0 \right\}.$$

Proof: Injectivity: lecture 16, surjectivity: this lecture.

A caution: CTeich **is smooth, but non-Hausdorff.** The non-Hausdorff points are well understood and correspond to the partition of the "positive cone" $\{v \in H^{1,1}_I(M, \mathbb{R}) \quad | \quad \int_M v \wedge v > 0\}$ onto "Kähler chambers" (to be explained later).

C-symplectic structures on surfaces (reminder)

CLAIM: In real dimension 4, C-symplectic structure **is determined by a pair $\omega_1 = \operatorname{Re}\Omega, \omega_2 = \operatorname{Im}\Omega$ of symplectic forms which satisfy $\omega_1^2 = \omega_2^2$ and $\omega_1 \wedge \omega_2 = 0$.**

Proof: Let Ω be a C-symplectic form, $\omega_1 = \operatorname{Re}\Omega$ and $\omega_2 = \operatorname{Im}\Omega$. Then $\Omega \wedge \Omega = \omega_1^2 + \omega_2^2 + 2\sqrt{-1}\,\omega_1 \wedge \omega_2 = 0$, hence $\omega_1^2 = \omega_2^2$ and $\omega_1 \wedge \omega_2 = 0$. The form $\Omega \wedge \overline{\Omega} = \omega_1^2 + \omega_2^2$ is non-degenerate, hence $\omega_1^2 = \omega_2^2$ is non-degenerate.

Conversely, if $\omega_1^2 = \omega_2^2$ and $\omega_1 \wedge \omega_2 = 0$, we have $\Omega \wedge \Omega = 0$, and $\Omega \wedge \overline{\Omega} = \omega_1^2 + \omega_2^2$ is non-degenerate if ω_i is non-degenerate. ■

REMARK: For K3 surface, the local Torelli theorem is equivalent to the following statement: the period map $\operatorname{Per}: \operatorname{CTeich} \longrightarrow H^2(M, \mathbb{C})$ which takes ω_1, ω_2 to their cohomology classes which satisfy $[\omega_1]^2 = [\omega_2]^2$ and $[\omega_1] \wedge [\omega_2] = 0$ **is locally a diffeomorphism.**

dd^c-lemma

THEOREM: Let η be a form on a compact Kähler manifold, satisfying one of the following conditions.
(1). η is an exact (p,q)-form. (2). η is d-exact, d^c-closed.
(3). η is ∂-exact, $\bar\partial$-closed.
Then $\eta \in \operatorname{im} dd^c = \operatorname{im} \partial\bar\partial$.

Proof: Notice immediately that in all three cases η **is closed and orthogonal to the kernel of Δ**, hence its cohomology class vanishes. Indeed, $\ker\Delta$ is orthogonal to the image of $\partial, \bar\partial$ and d. Since η is exact, it lies in the image of Δ. Operator $G_\Delta := \Delta^{-1}$ is defined on $\operatorname{im}\Delta = \ker\Delta^\perp$ and commutes with d, d^c.

In case (1), η is d-exact, and $I(\eta) = (\sqrt{-1}\,)^{p-q}\eta$ is d-closed, hence η is d-exact, d^c-closed like in (2). Then $\eta = d\alpha$, where $\alpha := G_\Delta d^*\eta$. Since G_Δ and d^* commute with d^c, the form α is d^c-closed; since it belongs to $\operatorname{im}\Delta = \operatorname{im} G_\Delta$, it is d^c-exact, $\alpha = d^c\beta$ which gives $\eta = dd^c\beta$.

In case (3), we have $\eta = \partial\alpha$, where $\alpha := G_\Delta\partial^*\eta$. Since G_Δ and ∂^* commute with $\bar\partial$, the form α is $\bar\partial$-closed; since it belongs to $\operatorname{im}\Delta$, it is $\bar\partial$-exact, $\alpha = \bar\partial\beta$ which gives $\eta = \partial\bar\partial\beta$. ■

Massey products

As an application of dd^c-lemma, I would prove a theorem about topology of compact Kähler manifolds.

Let $a, b, c \in \Lambda^*(M)$ be closed forms on a manifold M with cohomology classes $[a], [b], [c]$ satisfying $[a][b] = [b][c] = 0$, and $\alpha, \gamma \in \Lambda^*(M)$ forms which satisfy $d(\alpha) = a \wedge b$, $d(\gamma) = b \wedge c$. Denote by $L_{[a]}, L_{[c]} : H^*(M) \longrightarrow H^*(M)$ the operation of multiplication by the cohomology classes $[a], [c]$.

Then $\alpha \wedge c - a \wedge \gamma$ is a closed form, and its cohomology class is well-defined modulo $\operatorname{im} L_{[a]} + \operatorname{im} L_{[c]}$.

DEFINITION: Cohomology class $\alpha \wedge c - a \wedge \gamma$ is called **Massey product of a, b, c.**

PROPOSITION: On a Kähler manifold, Massey products vanish.

Proof: Let a, b, c be harmonic forms of pure Hodge type, that is, of type (p, q) for some p, q. Then ab and bc are exact pure forms, hence $ab, bc \in \operatorname{im} dd^c$ by dd^c-lemma. This implies that $\alpha := d^* G_\Delta(ab)$ and $\gamma := d^* G_\Delta(bc)$ are d^c-exact. Therefore $\mu := \alpha \wedge c - a \wedge \gamma$ is a d^c-exact, d-closed form. **Applying dd^c-lemma again, we obtain that μ is dd^c-exact, hence its cohomology class vanish.**

■

Heisenberg group

REMARK: *In the class, we constructed this space explicitly as a cell complex, without using the Lie algebra, and computed the Massey product in its cohomology. Here are the notes taken from a lecture given elsewhere, which explain the same construction in a different, more algebraic way.*

DEFINITION: The **Heisenberg group** G group of strictly upper triangular matrices (3x3),

$$\begin{pmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix}$$

The integer Heisenberg group $G_{\mathbb Z}$ is the same group with integer entries. The **Heisenberg nilmanifold** is $G/G_{\mathbb Z}$. The Heisenberg nilmanifold **is fibered over the torus T^2 with the fiber S^1** (it is a non-trivial principal S^1-bundle). This fibration corresponds to the exact sequence

$$\{e\} \longrightarrow \mathbb Z \longrightarrow G_{\mathbb Z} \longrightarrow \mathbb Z^2 \longrightarrow \{e\}$$

where $\mathbb Z$ is the center.

Massey products in Heisenberg nilmanifold

CLAIM: Masey products on $G/G_{\mathbb{Z}}$ are non-zero.

Proof. Step 1: G acts on $\Lambda^*(G)$ from the right. It is not hard to see that the all cohomology classes on $G/G_{\mathbb{Z}}$ can be represented by right G-invariant forms, and, moreover, **the cohomology of $G/G_{\mathbb{Z}}$ is equal to the cohomology of the complex of right-G-invariant forms on G.**

Step 2: This is the same complex as **the Chevalley-Eilenberg complex** for the Lie algebra $\mathfrak{g}$ of G: $0 \longrightarrow \Lambda^1(\mathfrak{g}^*) \xrightarrow{d} \Lambda^2(\mathfrak{g}^*) \xrightarrow{d} ...$ with the differential in the first term $d: \mathfrak{g}^* \longrightarrow \Lambda^2(\mathfrak{g}^*)$ dual to the commutator. We extend this differential to $\Lambda^*(\mathfrak{g}^*)$ by the Leibniz rule. The corresponding cohomology is called **the Lie algebra cohomology** and denoted by $H^*(\mathfrak{g})$.

Step 3: Let a, b, t be the basis in $\mathfrak{g}$, with the only non-trivial commutator $[a, b] = t$, and α, β, τ the dual basis in $\mathfrak{g}^*$, with the only non-trivial differential $d\tau = \alpha \wedge \beta$. This gives a basis $\alpha \wedge \beta$, $\alpha \wedge \tau$, $\beta \wedge \tau$ in $\Lambda^2(\mathfrak{g}^*)$, with $d\Big|_{\Lambda^2\mathfrak{g}^*} = 0$, giving $\operatorname{rk} H^1(G/G_{\mathbb{Z}}) = 2$ and $\operatorname{rk} H^2(G/G_{\mathbb{Z}}) = 2$.

Step 4: Let $M(\alpha, \beta, \alpha)$ denote the Massey product of α, β, α. Since $\alpha \wedge \beta = d\tau$, $M(\alpha, \beta, \alpha) = \tau \wedge \alpha - \alpha \wedge \tau = 2\tau \wedge \alpha$. The image of $L_\alpha: H^1(\mathfrak{g}) \longrightarrow H^2(\mathfrak{g})$ is generated by $\alpha \wedge \beta$, **hence $M(\alpha, \beta, \alpha)$ is non-zero modulo $\operatorname{im} L_\alpha$.** ■

Local Torelli theorem: surjectivity of the period map

THEOREM: Let (M, I, Ω) be a compact complex holomorphically symplectic surface. Then for any sufficiently small cohomology class $[\eta] \in H^{1,1}(M)$, **there exists a C-symplectic form $\Omega_\eta := \Omega + \rho$, where $\rho \in \Lambda^{1,1}M + \Lambda^{0,2}M$ is a closed form which satisfies $\rho^{1,1} \wedge \rho^{1,1} = -\Omega \wedge \rho^{0,2}$, and $\rho^{1,1}$ is ∂-cohomologous to $[\eta]$.**

Proof: Later today

REMARK: Clearly, the cohomology class $[u]$ of $\Omega+\rho$ is equal to $[\Omega+\eta+u^{0,2}]$. Since M is K3, we have $H^{0,2}(M) = \mathbb{C}[\overline{\Omega}]$, which gives $[u^{0,2}] = \lambda[\overline{\Omega}]$, for some $\lambda \in \mathbb{C}$. Since $(\Omega + \rho)^2 = 0$, this gives $[\Omega \wedge u^{0,2}] = [\eta]$. Then $\lambda = -\frac{[\eta^2]}{[\Omega\wedge\overline{\Omega}]}$. **This implies that the cohomology class of $\Omega_\eta := \Omega + \rho$ is equal to $\Omega + \eta - \frac{[\eta^2]}{[\Omega\wedge\overline{\Omega}]}\overline{\Omega}$.**

REMARK: This theorem proves that the period map is surjective to a codimension 1 subvariety in $H^2(M, \mathbb{C})$. Together with injectivity, proven in lecture 16, **this implies the local Torelli theorem for K3.**

Local Torelli theorem for K3 (2)

THEOREM: Let (M, I, Ω) be a complex holomorphically symplectic surface with $H^{0,1}(M) = 0$, that is, a K3 surface. Then for any sufficiently small cohomology class $[\eta] \in H^{1,1}(M)$, **there exists a C-symplectic form $\Omega_\eta := \Omega+\rho$, where $\rho \in \Lambda^{1,1}M+\Lambda^{0,2}M$ is a closed form which satisfies $\rho^{1,1}\wedge\rho^{1,1} = -\Omega \wedge \rho^{0,2}$, and $\rho^{1,1}$ is ∂-cohomologous to $[\eta]$.**

REMARK: We write ρ as a Taylor series depending on $\eta \in H^{1,1}(M, \mathbb{C})$; for η sufficiently small, this term is small, and $\Omega_\eta := \Omega+\rho$ remains non-degenerate. **The only non-trivial condition to check is $d\Omega_\eta = 0$.**

REMARK: In the next slide, we need the following version of $\partial\bar\partial$**-lemma**: **for any $(1,2)$-form α, which is ∂-exact and $\bar\partial$-closed, $\alpha = \bar\partial\beta$, where β is ∂-exact.**

Local Torelli theorem for K3 (3)

Proof. Step 1: Let Λ_Ω be contraction with the $(2,0)$-bivector associated with Ω. This operation clearly commutes with $\overline{\partial}$, and is inverse to an isomorphism taking $\Lambda^{0,2}(M) \xrightarrow{\wedge\Omega} \Lambda^{2,2}(M)$. Then $\rho^{1,1}\wedge\rho^{1,1} = -\Omega\wedge\rho^{0,2}$ is equivalent to $\Lambda_\Omega(\rho^{1,1}\wedge\rho^{1,1}) = -\rho^{0,2}$. **To solve the equation $d\rho = 0$, we solve the equivalent equation:**

$$\partial\Lambda_\Omega(\rho^{1,1}\wedge\rho^{1,1}) = -\overline{\partial}\rho^{1,1}, \qquad \partial\rho^{1,1} = 0 \qquad (*)$$

Let γ_0 be the harmonic (1,1)-form representing $[\eta]$. We solve the equation (*) inductively by taking

$$\overline{\partial}\gamma_n = \partial\Lambda_\Omega\left(\sum_{i+j=n-1}\gamma_i\wedge\gamma_j\right), \quad \gamma_n \text{ is } \partial\text{-exact} \quad (**)$$

Such $\gamma_n \in \Lambda^{1,1}(M,I)$ is found using $\partial\overline{\partial}$-lemma, because the RHS of (**) is ∂-exact and $\overline{\partial}$-closed. The latter is clear because $\overline{\partial}$ commutes with Λ_Ω, and the (2,2)-forms $\gamma_i\wedge\gamma_j$ are clearly $\overline{\partial}$-closed. Since $\overline{\partial}\sum_i\gamma_i = \partial\Lambda_\Omega\left(\sum_{i,j}\gamma_i\wedge\gamma_j\right)$, the sum $\rho^{1,1} := \sum\gamma_i$ is a solution of (*).

Step 2: Since γ_i, $i>0$ are ∂-exact, the ∂-cohomology class of $\sum\gamma_i$ is $[\gamma_0] = [\eta]$. **This proves the claim of the theorem, conditional on convergence of the series $\sum\gamma_i$,** which is explained in the next two slides. ∎

Convergence of the solutions of the Mauer-Cartan equation

In the previous slide, we wrote a recursive solution $\rho^{1,1} = \sum_i \gamma_i$ of the equation

$$\partial\Lambda_\Omega(\rho^{1,1}\wedge\rho^{1,1}) = -\overline{\partial}\rho^{1,1}, \qquad \partial\rho^{1,1} = 0 \qquad (*)$$

which is given by

$$\overline{\partial}\gamma_n = \partial\Lambda_\Omega\left(\sum_{i+j=n-1}\gamma_i\wedge\gamma_j\right), \quad \gamma_n \text{ is } \partial\text{-exact} \quad (**)$$

It remains to prove its convergence.

Let G_Δ be the Green operator inverting the Laplacian $\Delta = \overline{\partial}\overline{\partial}^* + \overline{\partial}^*\overline{\partial}$ on forms which are orthogonal to harmonic forms. **Then $G_{\overline{\partial}} := \overline{\partial}^* G_\Delta$ inverts $\overline{\partial}$ on $\overline{\partial}$-exact forms.** From Hodge theory it follows easily that $\Psi(x) := G_{\overline{\partial}}\partial\Lambda_\Omega(x)$ is continuous. Let $K := \|\Psi\|$ be its operator norm.

DEFINITION: The n-th Catalan number is defined as the number of distinct ways one can put n pair of parentheses in a word on $n+1$ letters. For example, for a word $abcd$, **there are exactly 5 ways to put 3 pairs of parentheses: $((ab)(cd))$, $((a(bc))d)$, $(a((bc)d))$, $(((ab)c)d)$, $(a(b(cd)))$.**

Convergence of the solutions of the Mauer-Cartan equation (2)

Recursive solution of Maurer-Cartan: $\overline{\partial}\gamma_n = \partial\Lambda_\Omega\left(\sum_{i+j=n-1}\gamma_i\wedge\gamma_j\right) \quad (**).$

CLAIM: Let $\gamma_n := G_{\overline{\partial}}\partial\Lambda_\Omega\left(\sum_{i+j=n-1}\gamma_i\wedge\gamma_j\right) = \Psi\left(\sum_{i+j=n-1}\gamma_i\wedge\gamma_j\right)$ be solutions of (**), obtained by inverting $\overline{\partial}$ through the Green operators. **Then $|\gamma_n| \leqslant C_n|\gamma_0|^{n+1}|K|^n$, where C_n is the n-th Catalan number.**

Proof: If we open all brackets, we obtain that γ_n **is a sum of C_n terms obtained by putting n parentheses in a word $\underbrace{\gamma_0\gamma_0......\gamma_0\gamma_0}_{n+1 \text{ times}}$, with each parenthesis encoding the expression $\Psi(...)$ and all consecutive terms wedge-multiplied.** For example, the term $((a(bc))d)$ would correspond to $\Psi(\Psi(\gamma_0\wedge\Psi(\gamma_0\wedge\gamma_0))\wedge\gamma_0)$. Each of these terms is clearly bounded by $|\gamma_0|^{n+1}|K|^n$ ■

To prove the convergence of $\sum\gamma_i$, it remains to estimate $C_n = \frac{1}{n+1}\binom{2n}{n}$; Stirling formula easily implies that $C_n = \frac{4^n}{\sqrt{\pi n^3}}(1+O(1/n))$, hence

$$\gamma_n \leqslant 4^n K^n|\gamma_0|^{n+1}(1+O(1/n)),$$

and **it decays faster than a geometric progression once $|\gamma_0|^{-1} > 4K$.**