K3 surfaces

lecture 18: Kummer surfaces

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Total space of $T^*\mathbb{C}P^1$ is a resolution of $\mathbb{C}^2/\{\pm 1\}$

EXERCISE: Let $X \stackrel{\pi}{\longrightarrow}$ be a blow-up of $\mathbb{C}^2$ in a point. **Prove that X is isomorphic to the total space of $\mathcal{O}(-1)$ on $\mathbb{C}P^1$.**

CLAIM: The total space of the holomorphic line bundle $T^*\mathbb{C}P^1 = \mathcal{O}(-2)$ **is isomorphic to a blow-up of $\mathbb{C}^2/\{\pm 1\}$ in 0.**

Proof: Consider a ramified covering $\operatorname{Tot}(\mathcal{O}(-1)) \longrightarrow \operatorname{Tot}(T^*\mathbb{C}P^1)$ taking a vector $v \in \mathcal{O}(-1)|_s$ to $v \otimes v \mathcal{O}(-2)|_s$. It defines a commutative square

$$\begin{array}{ccc} \operatorname{Tot}(\mathcal{O}(-1)) & \longrightarrow & \operatorname{Tot}(T^*\mathbb{C}P^1) \\ \Big\downarrow & & \Big\downarrow \\ \mathbb{C}^2 & \longrightarrow & \mathbb{C}^2/\{\pm 1\}. \end{array}$$

The horizontal arrows of this diagram are 2:1 ramified covers, and the vertical arrows are birational. ■

EXERCISE: Prove that the space $\mathbb{C}^2/\{\pm 1\}$, considered as an affine complex variety, **is the spectrum of an affine ring $\frac{\mathbb{C}[x,y,z]}{(z^2 - xy = 0)}$.** Prove that its blow-up is $\operatorname{Tot}(T^*\mathbb{C}P^1)$.

REMARK: Clearly, the total space $\operatorname{Tot}(T^*\mathbb{C}P^1)$ is holomorphically symplectic, and the blow-up map $\operatorname{Tot}(T^*\mathbb{C}P^1) \longrightarrow \mathbb{C}^2/\{\pm 1\}$ **takes this symplectic form to the constant holomorphic symplectic form on $\mathbb{C}^2/\{\pm 1\}$.**

Kummer surface

DEFINITION: Let T^2 be a 2-dimensional compact complex torus. If we fix the origin, we can consider T^2 as a complex abelian Lie group. Consider an involution of T^2 taking $x \in T^2$ to $-x$. **This involution has 16 fixed points** (2-torsion point are points which satisfy $x = -x$), and in a neighbourhood of each fixed point, $T^2/\{\pm 1\}$ is symplectomorphic to $\mathbb{C}^2/\{\pm 1\}$. Using the previous remark, **we obtain that the blow-up of $T^2/\{\pm 1\}$ in 16 singular points is holomorphically symplectic.** This surface is called **a Kummer surface**.

PROPOSITION: A Kummer surface K is of K3 type.

Proof. Step 1: The holomorphic symplectic form on T^2 is $\{\pm 1\}$-invariant, and extends to the blow-up as shown on the previous slide. **It remains only to show that $\pi_1(K) = 0$.**

Kummer surface (2)

Step 2: The space $(S^1 \times S^1)/\pm 1$ is homeomorphic to the sphere S^2. Indeed, a quotient map $(S^1 \times S^1) \longrightarrow (S^1 \times S^1)/\pm 1$ takes an elliptic curve in $\mathbb{C}P^2$ to the space of all lines passing through its zero, and this space is by construction homeomorphic (and biholomorphic) to $\mathbb{C}P^1$.

Step 3: This implies that $T^2/\pm 1 = (S^1)^4/\pm 1$ is simply connected. Indeed, any path in $(S^1)^4$ is homotopic to a composition of paths which are contained in some S^1, and the image of all such paths in $\pi_1((S^1)^4/\pm 1)$ is trivial by Step 2.

Step 4: Let $W_1, ...W_{16} \subset T^2/\pm 1$ be neighbourhoods of the 16 singular points, chosen in such a way that their resolutions $\tilde{W}_1, ..., \tilde{W}_{16} \subset K$ are 2ε-neighbourhoods of 16 exceptional $\mathbb{C}P^1$ in K. Let K_0 be the complement to the union of ε-neighbourhoods of these 16 curves. By Seifert-van Kampen theorem, $\pi_1(K)$ is a free product of $\pi_1(\tilde{W}_i)$ and $\pi_1(K_0)$ under the identification which identifies the images of $\pi_1(\tilde{W}_i \cap K_0)$ in $\pi_1(\tilde{W}_i)$ and $\pi_1(K_0)$. Similarly, $\pi_1((S^1)^4/\pm 1)$ is a free product of $\pi_1(W_i)$ and $\pi_1(K_0)$ under the identification which identifies the images of $\pi_1(W_i \cap K_0)$ in $\pi_1(W_i)$ and $\pi_1(K_0)$. **Since all W_i and $\tilde{W}_i$ are simply connected, we have a natural isomorphism $\pi_1(K) = \pi_1(T^2/\pm) = 0$.** ■