K3 surfaces

lecture 19: (-2)-curves

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Effective cohomology classes

DEFINITION: Let $\eta \in H^{1,1}(M,\mathbb{Z})$ be an integer (1,1)-class on a surface. It is called **effective** if it is a fundamental class of a divisor.

CLAIM: Let M be a K3 surface, and $\eta \in H^{1,1}(M,\mathbb{Z})$ a cohomology class such that $\int_M \eta\wedge\eta \geqslant -2$. **Then η or $-\eta$ is effective.**
Proof: Let L be a line bundle such that $c_1(L)=\eta$. Riemann-Roch formula gives $\chi(L) = 2+\frac{(L,L)}{2} \geqslant 1$. This implies that either $\dim H^0(L) > 0$ or $\dim H^2(L) > 0$. In the first case, η is the fundamental class of the zero divisor of a holomorphic section of L. In the second case, we use Serre's duality to obtain that $H^2(L)$ is dual to $H^0(L^*\otimes K_M) = H^0(L^*)$. If $\dim H^2(L) = \dim H^0(L^*)$, then $-\eta$ is the fundamental class of the zero divisor of a holomorphic section of L^*. ■

What about other integer classes in $H^{1,1}(M)$?
THEOREM: Let $S\subset M$ be a connected curve on a K3 surface. **Then $(S,S)\geqslant -2$.**
Proof: Later today

REMARK: This result **does not imply that a class $\eta\in H^{1,1}(M,\mathbb{Z})$ with $\int_M \eta\wedge\eta < -2$ cannot be effective.** Indeed, the Kummer surface contains two (-2)-curves S_1, S_2 which do not intersect, and their union $S := S_1\cup S_2$ satisfies $(S,S) = -4$.

Riemann-Roch theorem for curves in a K3 surface

Lemma 1: Let $C \subset M$ be a complex curve in a K3 surface. **Then** $\chi(\mathcal{O}_C) = -\frac{(C,C)}{2}$.

Proof: Consider the exact sequence $0 \longrightarrow \mathcal{O}_M(-C) \longrightarrow \mathcal{O}_M \longrightarrow \mathcal{O}_C \longrightarrow 0$. It gives $\chi(\mathcal{O}_C) = \chi(\mathcal{O}_M) - \chi(\mathcal{O}_M(-C))$. Using $\chi(L) = 2 + \frac{(L,L)}{2}$ again, we obtain that $\chi(\mathcal{O}_C) = 2 - 2 - \frac{(C,C)}{2} = -\frac{(C,C)}{2}$ ■

Corollary 1: This implies that **if C is smooth, we have** $g(C) = \frac{(C,C)}{2} + 1$.

Proof: Indeed, $\chi(\mathcal{O}_C) = 1 - g(C)$. ■

Irreducible curves on K3

Proposition 1: Let $S \subset M$ be an irreducible curve on a K3 surface. **Then** $(S,S) \geqslant -2$. This inequality is strict **unless S is a smooth rational curve.**

Proof. Step 1: Let $\tilde{S} \xrightarrow{\varphi} S$ be the normalization of S; it is a smooth compact complex curve, and the map φ is finite. Consider an exact sequence

$$0 \longrightarrow \mathcal{O}_S \longrightarrow \varphi_* \mathcal{O}_{\tilde{S}} \longrightarrow K \longrightarrow 0. \quad (*)$$

By construction, K is a coherent sheaf with support in singularities of S. Since φ is finite, φ_* commutes with cohomology (Lecture 9), which gives $\chi(\varphi_* \mathcal{O}_{\tilde{S}}) = \chi(\mathcal{O}_{\tilde{S}})$. **The exact sequence (*) then gives $\chi(\mathcal{O}_S) + \chi(K) = \chi(\varphi_* \mathcal{O}_{\tilde{S}}) = \chi(\mathcal{O}_{\tilde{S}})$.**

Step 2: Let g be the genus of $\tilde{S}$. Then $\chi(\mathcal{O}_S) = -\frac{(S,S)}{2}$ (Lemma 1); Step 1 gives

$$-\frac{(S,S)}{2} = \chi(\mathcal{O}_S) = \chi(\mathcal{O}_{\tilde{S}}) - \chi(K) = 1 - g - \chi(K),$$

equivalently, $\frac{(S,S)}{2} + 1 = g + \chi(K)$. Since K is a torsion sheaf on S, $\chi(K) \geqslant 0$. **Then $\frac{(S,S)}{2} \leqslant -2$ implies that $g < 0$, which is impossible.** If $(S,S) = -2$, we have $\frac{(S,S)}{2} + 1 = 0$, which implies $g = 0$ and $\chi(K) = 0$, hence S is a smooth curve of genus zero. ■

(-2)-curves

DEFINITION: A **(-2)-curve** on a K3 surface is a connected curve C such that $(C,C)=-2$.

Theorem 1: Any (-2)-curve **is a collection of p smooth (-2)-curves which intersect transversally in $p-1$ points** (hence, **their incidence graph is a tree).**

Proof. Step 1: Let $C_1, ... C_n$ be irreducible components of C. Then $-2 = (C,C) = \sum_i (C_i, C_i) + 2\sum_{i\neq j}(C_i, C_j)$. For each distinct pair of curves wich intersect, we have $(C_i, C_j) \geqslant 1$. This gives $-2 = \sum_i (C_i, C_i) + 2m$, where m is the number of intersections taken with multiplicities, and $p = \sum_i (C_i, C_i)$. By Proposition 1, $(C_i, C_i) \geqslant -2$, and the equality is only realized when C_i is smooth and has genus 0. Therefore, $p \geqslant -2n$, with equality only realized when all C_i are smooth and have genus 0.

Step 2: Since C is connected, there are at least $n-1$ pairwise intersections. this implies $m \geqslant (n-1)$, and the inequality is strict unless all intersections are transversal. Then $-2 = p + 2m \geqslant -2n + 2(n-1)$, which can happen only if both inequalities $p \geqslant -2n$ and $m \geqslant (n-1)$ are non-strict; in other words, there is a connected set of n smooth rational curves, intersecting transversely. ■

(-2)-curves are contractible

REMARK: From Theorem 1, it follows that the incidence graph of the curves C_i is a tree. In fact, it is possible to show that **it is always the Dynkin graph for the root systems A_i, D_i, E_6, E_7, E_8.** It is implied by du Val's classification of canonical singularities of complex surfaces, and the following theorem

THEOREM: Let $C \subset M$ be a (-2)-curve on a K3 surface. **Then there exists a holomorphic, bimeromorphic map $\Psi: M \longrightarrow M_1$ to a singular complex variety, contracting C to a point, and bijective outside of C.** If, moreover, M is projective, **then M_1 is also projective, and Ψ is algebraic.**

REMARK: I will prove only the second part; the first is Grauert's contractibility theorem, *Grauert, H.: Uber Modifikationen und exzeptionelle analytischen Mengen, Math. Ann. 146 (1962), 331-368,* see also *Akira Fujiki, On the Blowing Down of Analytic Spaces, Publ. RIMS, 1975 Volume 10 Issue 2 Pages 473-507,* https://www.jstage.jst.go.jp/article/kyotoms1969/10/2/10_2_473/_article. Also, I will assume that C is irreducible; to prove it in full generality, we would need to use induction on the number of irreducible components of C, and show that each successive contraction results in a "K3-type orbifold".

(-2)-curves are contractible (2)

THEOREM: Let $C \subset M$ be an irreducible (-2)-curve on a projective K3 surface. **Then there exists a holomorphic, bimeromorphic map $\Psi : M \longrightarrow M_1$ to a complex projective variety, contracting C to a point, and bijective outside of C.**

Proof. Step 1: Let L be the Chern class of an ample line bundle. and C a (-2)-class, $(L, C) = k$. Then $(2L + kC, 2L + kC) = 4(L, L) + 4k(L, C) - 2k^2 = 4(L, L) + 4(L, C)^2 - 2(L, C)^2 = 4(L, L) + 2(L, C)^2$, and $(2L + kC, C) = 2k - 2k = 0$. Let L_k be the line bundle with $c_1(L_k) = 2L + kC$. **We are going to show that the natural map $M \longrightarrow \mathbb{P}H^0(M, L_k)^*$ is holomorphic, contracts C, and is injective on its complement,** if L is very ample.

Step 2: Let $x \in [0, k]$, and $L_x := 2L + xC$. Since the class $2L + xC$ has positive square for all $x \in [0, k]$, and $2L$ belongs to the positive cone, L_x belongs to the positive cone in $H^{1,1}(M, \mathbb{R})$. By Hodge index formula, it is positive on all curves S such that $(S, S) \geqslant 0$. Given an irreducible (-2)-curve C_1 which does not coincide with C, we have $(L_x, C_1) = 2(L, C_1) + k(C, C_1) > 0$. Finally, $(L_x, C) = 2k - 2x$. This implies that L_x **is positive on any irreducible curve except C;** also **it is positive on C if $0 \leqslant x < k$.** Nakai-Moishezon theorem **implies that L_x is ample when $0 \leqslant x \leqslant k - 1$.**

(-2)-curves are contractible (3)

Step 3: Since $(L_k, C) = 0$, we have $L_k|_C = \mathcal{O}_C$. This gives an exact sequence

$$0 \longrightarrow L_{k-1} \longrightarrow L_k \longrightarrow \mathcal{O}_C \longrightarrow 0 \quad (*)$$

Since $K_M = 0$ and L_{k-1} is ample (Step 2), Kodaira-Nakano vanishing theorem implies that $H^i(L_{k-1}) = 0$ for all $i > 0$, hence (*) **gives an exact sequence**

$$0 \longrightarrow H^0(L_{k-1}) \longrightarrow H^0(L_k) \longrightarrow H^0(\mathcal{O}_C) = \mathbb{C} \longrightarrow 0.$$

It remains to show that L_{k-1} is very ample, when L is very ample.

Step 4: As in Step 2, consider the bundle $L_i = L^{\otimes 2} \otimes \mathcal{O}(iC)$, $i < k-1$; we can freely assume $L_0 = L^{\otimes 2}$ is very ample. To finish the proof of the theorem, **it would suffice to show that L_{k-1} is very ample (Step 3).** We use induction in i; suppose that L_i is very ample. The exact sequence

$$0 \longrightarrow L_i \longrightarrow L_{i+1} = L_i \otimes \mathcal{O}(C) \longrightarrow L_{i+1}|_C \longrightarrow 0$$

and ampleness of L_i gives an exact sequence

$$0 \longrightarrow H^0(L_i) \longrightarrow H^0(L_{i+1}) \longrightarrow H^0(L_{i+1}|_C) \longrightarrow 0. \quad (**)$$

Since $i < k-1$, the bundle $H^0(L_{i+1})$ has degree $\geqslant 2$ on C, which is smooth and rational, and therefore $L_{i+1}|_C$ is very ample. Then the exact sequence (**) implies that L_{i+1} is very ample a well. ■