# K3 surfaces

## lecture 20: Calabi-Yau theorem and hyperkähler structures

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## Holomorphic vector bundles

**DEFINITION: A $\overline{\partial}$-operator** on a smooth bundle is a map $V \xrightarrow{\overline{\partial}} \Lambda^{0,1}(M)\otimes V$, satisfying $\overline{\partial}(fb) = \overline{\partial}(f)\otimes b + f\overline{\partial}(b)$ for all $f \in C^\infty M, b \in V$.

**REMARK: A $\overline{\partial}$-operator on $B$ can be extended to**

$$\overline{\partial}: \Lambda^{0,i}(M)\otimes V \longrightarrow \Lambda^{0,i+1}(M)\otimes V,$$

using $\overline{\partial}(\eta\otimes b) = \overline{\partial}(\eta)\otimes b + (-1)^{\tilde{\eta}}\eta\wedge\overline{\partial}(b)$, where $b \in V$ and $\eta \in \Lambda^{0,i}(M)$.

**DEFINITION: A holomorphic vector bundle** on a complex manifold $(M, I)$ is a vector bundle equipped with a $\overline{\partial}$-operator which satisfies $\overline{\partial}^2 = 0$. In this case, $\overline{\partial}$ is called **a holomorphic structure operator**.

**EXERCISE:** Consider the Dolbeault differential $\overline{\partial}: \Lambda^{p,0}(M) \longrightarrow \Lambda^{p,1}(M) = \Lambda^{p,0}(M)\otimes\Lambda^{0,1}(M)$. **Prove that it is a holomorphic structure operator on $\Lambda^{p,0}(M)$.**

**DEFINITION:** The corresponding holomorphic vector bundle $(\Lambda^{p,0}(M), \overline{\partial})$ is called **the bundle of holomorphic $p$-forms**, denoted by $\Omega^p(M)$.

## Chern connection

**DEFINITION:** Let $(B,\nabla)$ be a smooth bundle with connection and a holomorphic structure $\overline{\partial}\ B \longrightarrow \Lambda^{0,1}(M)\otimes B$. Consider a Hodge decomposition of $\nabla$, $\nabla=\nabla^{0,1}+\nabla^{1,0}$,

$$\nabla^{0,1}:\ V \longrightarrow \Lambda^{0,1}(M)\otimes V, \quad \nabla^{1,0}:\ V \longrightarrow \Lambda^{1,0}(M)\otimes V.$$

We say that $\nabla$ is **compatible with the holomorphic structure** if $\nabla^{0,1}=\overline{\partial}$.

**DEFINITION: An Hermitian holomorphic vector bundle** is a smooth complex vector bundle equipped with a Hermitian metric and a holomorphic structure operator $\overline{\partial}$.

**DEFINITION: A Chern connection** on a holomorphic Hermitian vector bundle is a connection compatible with the holomorphic structure and preserving the metric.

**THEOREM:** On any holomorphic Hermitian vector bundle, **the Chern connection exists, and is unique.**

## Curvature of a connection

**DEFINITION:** Let $\nabla : B \longrightarrow B \otimes \Lambda^1 M$ be a connection on a smooth budnle. Extend it to an operator on $B$-valued forms

$$B \xrightarrow{\nabla} \Lambda^1(M) \otimes B \xrightarrow{\nabla} \Lambda^2(M) \otimes B \xrightarrow{\nabla} \Lambda^3(M) \otimes B \xrightarrow{\nabla} ...$$

using $\nabla(\eta \otimes b) = d\eta + (-1)^{\tilde\eta}\eta \wedge \nabla b$. The operator $\nabla^2 : B \longrightarrow B \otimes \Lambda^2(M)$ is called **the curvature** of $\nabla$. The operator $\nabla : \Lambda^i(M) \otimes B \xrightarrow{\nabla} \Lambda^{i+1}(M) \otimes B$ **is often denoted $d_\nabla$.**

**REMARK:** The algebra of $\operatorname{End}(B)$-valued forms naturally acts on $\Lambda^* M \otimes B$. The curvature satisfies $\nabla^2(fb) = d^2 fb + df \wedge \nabla b - df \wedge \nabla b + f\nabla^2 b = f\nabla^2 b$, hence it is $C^\infty M$-linear. **We consider it as an $\operatorname{End}(B)$-valued 2-form on $M$.**

**REMARK: (Bianchi identity)**
Super-Jacobi identity implies $[\nabla, \nabla^2] = [\nabla^2, \nabla] + [\nabla, \nabla^2] = 0$, hence $[\nabla, \nabla^2] = 0$. This gives **the Bianchi identity:** $d_\nabla(\Theta_B) = 0$, where $\Theta$ is considered as a section of $\Lambda^2(M) \otimes \operatorname{End}(B)$, and $d_\nabla : \Lambda^2(M) \otimes \operatorname{End}(B) \longrightarrow \Lambda^3(M) \otimes \operatorname{End}(B)$ the operator defined above.

## Curvature of a holomorphic line bundle

**REMARK:** If $B$ is a line bundle, $\operatorname{End} B$ is trivial, and **the curvature $\Theta_B$ of $B$ is a closed 2-form.**

**DEFINITION:** Let $\nabla$ be a unitary connection in a line bundle. The cohomology class $c_1(B) := \frac{\sqrt{-1}}{2\pi}[\Theta_B] \in H^2(M)$ is called **the real first Chern class of a line bunlde $B$**.

**An exercise:** Check that $c_1(B)$ is independent from a choice of $\nabla$.

**REMARK:** When speaking of a **"curvature of a holomorphic bundle"**, one usually means the curvature of a Chern connection.

**REMARK:** Let $B$ be a holomorphic Hermitian line bundle, and $b$ its non-degenerate holomorphic section. Denote by $\eta$ a (1,0)-form which satisfies $\nabla^{1,0}b = \eta \otimes b$. Then $d|b|^2 = \operatorname{Re} g(\nabla^{1,0}b, b) = \operatorname{Re} \eta |b|^2$. **This gives $\nabla^{1,0}b = \frac{\partial |b|^2}{|b|^2} b = 2\partial \log |b| b$.**

**REMARK:** Then $\Theta_B(b) = 2\bar\partial\partial \log |b| b$, **that is, $\Theta_B = -2\partial\bar\partial \log |b|$.**

**COROLLARY:** If $g' = e^{2f} g$ – two metrics on a holomorphic line bundle, $\Theta, \Theta'$ their curvatures, **one has $\Theta' - \Theta = -2\partial\bar\partial f$**

## Ricci curvature of a complex manifold

**CLAIM: For any holomorphic function $f$, $dd^c \log|f| = 0$.**

**Proof:** $\log|f| = \frac{1}{2}\log f + \log\overline{f}$, that is, a real part of a holomorphic function. However, $dd^c$ is proportional to $\partial\overline{\partial}$, hence it **vanishes on all holomorphic and on all antiholomorphic functions.** ■

**REMARK:** Let $(L, h)$ be a line bundle with Hermitian metric $h$. Choose two non-vanishing local holomorphic sections $l_1, l_2$ of $L$. Then $l_2 = ul_1$, where $u$ is an invertible holomorphic function. This gives $dd^c \log|l_1|^2 = dd^c \log|l_2|^2 + dd^c \log|u|^2$; **the last term vanishes by the previous claim.** Therefore, the 2-form $dd^c \log|l_1|^2$ is independent from the choice of a non-vanishing local section $l_1$. For this reason, **the curvature of $(L, h)$ form is often denoted as $dd^c \log h$.**

**DEFINITION:** Let $(M, I, \mathrm{Vol})$ be a complex $n$-manifold equipped with a volume form $\mathrm{Vol} \in \Lambda^{n,n}(M)$. Consider the following Hermitian form on the canonical bundle $K_M = \Lambda^{n,0}(M, I)$, $|\varphi|^2 = \frac{\varphi\wedge\overline{\varphi}}{\mathrm{Vol}}$. **The Ricci curvature** of $(M, I, \mathrm{Vol})$, often denoted as $dd^c \log \mathrm{Vol}$, is the curvature of $K_M$ considered as a Hermitian vector bundle. The manifold $(M, I, \mathrm{Vol})$ is called **Ricci-flat** if its Ricci curvature vanishes.

## Calabi-Yau theorem and Monge-Ampère equation

**REMARK:** Let $(M,\omega)$ be a Kähler $n$-fold, and $\Omega$ a non-degenerate section of $K(M)$, Then $|\Omega|^2 = \frac{\Omega\wedge\overline{\Omega}}{\omega^n}$. If $\omega_1$ is a new Kaehler metric on $(M,I)$, $h, h_1$ the associated metrics on $K(M)$, then $\frac{h}{h_1} = \frac{\omega_1^n}{\omega^n}$.

**REMARK:** For two metrics $\omega_1, \omega$ in the same Kähler class, one has $\omega_1 - \omega = dd^c\varphi$, **for some function** $\varphi$ ($dd^c$-lemma).

**COROLLARY:** Let $M$ be a Calabi-Yau manifold, $\omega$ its Kähler form, $\Omega$ a non-degenerate section of the canonical bundle. A metric $\omega_1 = \omega + \partial\overline{\partial}\varphi$ **is Ricci-flat if and only if** $(\omega + dd^c\varphi)^n = \omega^n e^f$, where $-2\partial\overline{\partial}f = \Theta_{K,\omega}$ **(such** $f$ **exists by** $\partial\overline{\partial}$**-lemma).**

**Proof. Step 1:** For $f$ such that $-2\partial\overline{\partial}f = \Theta_{K,\omega}$, the curvature of the metric $h \longrightarrow \frac{h\wedge\overline{h}}{\omega^n e^f}$ on $K_M$ is equal to $\Theta_{K,\omega} + 2\partial\overline{\partial}f = 0$.

**Proof. Step 2:** $\omega_1$ **is Ricci-flat if and only if the induced metric on** $K_M$ **is flat,** which is equivalent to $(\omega + dd^c\varphi)^n = \omega^n e^f$. ■

To find a Ricci-flat metric **it remains to solve an equation** $(\omega + dd^c\varphi)^n = \omega^n e^f$ **for a given** $f$**.**

## The complex Monge-Ampère equation

Let $M$ be a manifold with trivial canonical bundle. To find a Ricci-flat metric **it will suffice to solve an equation $(\omega + dd^c\varphi)^n = \omega^n e^f$ for a given $f$.**

**THEOREM:** (Calabi-Yau) Let $(M, \omega)$ be a compact Kaehler $n$-manifold, and $f$ any smooth function. **Then there exists a unique up to a constant function $\varphi$** such that $(\omega + \sqrt{-1}\,\partial\bar{\partial}\varphi)^n = Ae^f\omega^n$, where $A$ is a positive constant obtained from the formula $\int_M Ae^f\omega^n = \int_M \omega^n$.

**DEFINITION:**

$$(\omega + \sqrt{-1}\,\partial\bar{\partial}\varphi)^n = Ae^f\omega^n,$$

is called **the Monge-Ampere equation.**

## Uniqueness of solutions of complex Monge-Ampere equation

**PROPOSITION: (Calabi)**
A complex Monge-Ampere equation **has at most one solution,** up to a constant.

**Proof. Step 1:** Let $\omega_1, \omega_2$ be solutions of Monge-Ampere equation. Then $\omega_1^n = \omega_2^n$. By construction, one has $\omega_2 = \omega_1 + \sqrt{-1}\,\partial\bar\partial\psi$. **We need to show $\psi = const$.**

**Step 2:** $\omega_2 = \omega_1 + \sqrt{-1}\,\partial\bar\partial\psi$ gives

$$0 = (\omega_1 + \sqrt{-1}\,\partial\bar\partial\psi)^n - \omega_1^n = \sqrt{-1}\,\partial\bar\partial\psi \wedge \sum_{i=0}^{n-1} \omega_1^i \wedge \omega_2^{n-1-i}.$$

**Step 3:** Let $P := \sum_{i=0}^{n-1} \omega_1^i \wedge \omega_2^{n-1-i}$. This is a strictly positive $(n-1, n-1)$-form. **There exists a Hermitian form $\omega_3$ on $M$ such that $\omega_3^{n-1} = P$.**

**Step 4:** Since $\sqrt{-1}\,\partial\bar\partial\psi \wedge P = 0$, this gives $\psi\partial\bar\partial\psi \wedge P = 0$. Stokes' formula implies

$$0 = \int_M \psi \wedge \partial\bar\partial\psi \wedge P = -\int_M \partial\psi \wedge \bar\partial\psi \wedge P = -\int_M |\partial\psi|_3^2 \omega_3^n.$$

where $|\cdot|_3$ is the metric associated to $\omega_3$. **Therefore $\bar\partial\psi = 0$.** ■

## Bochner's vanishing

**THEOREM:** (Bochner vanishing theorem) On a compact Ricci-flat Calabi-Yau manifold, **any holomorphic $p$-form $\eta$ is parallel** with respect to the Levi-Civita connection: $\nabla(\eta) = 0$.

**REMARK:** Its proof is based on spinors: $\eta$ gives a harmonic spinor, and **on a Ricci-flat Riemannian spin manifold, any harmonic spinor is parallel.**

**DEFINITION:** A **holomorphic symplectic manifold** is a manifold admitting a non-degenerate, holomorphic symplectic form.

**COROLLARY:** Consider a Ricci-flat metric on compact holomorphic symplectic Kähler manifold. **Then the hololomorphically symplectic form is parallel.** ■

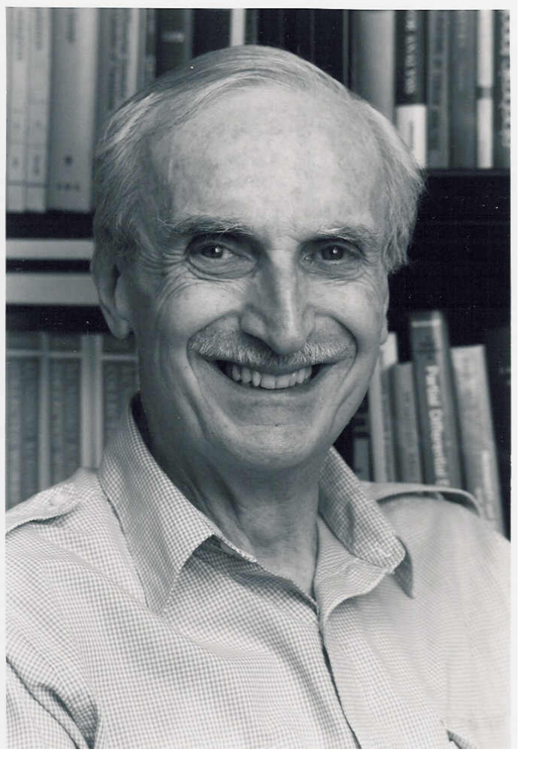
## Hyperkähler manifolds

**DEFINITION: (E. Calabi, 1978)**
Let $(M, g)$ be a Riemannian manifold equipped with three complex structure operators $I, J, K : TM \longrightarrow TM$, satisfying the quaternionic relations

$$I^2 = J^2 = K^2 = IJK = -\operatorname{Id}.$$

Suppose that $g$ is Kähler with respect to $I$, $J$, $K$. Then $(M, I, J, K, g)$ is called **hyperkähler**.

**REMARK: This is the same as $\mathcal{H}\text{ol}(M) \subset Sp(n)$.**
Indeed, if $\mathcal{H}\text{ol}(M) \subset Sp(n)$, we have 3 complex structures $I, J, K : TM \longrightarrow TM$, such that $\nabla(I) = \nabla(J) = \nabla(K) = 0$, which implies that $I, J, K$ are Kähler. Conversely, if $I, J, K$ are Kähler, we have $\nabla(I) = \nabla(J) = \nabla(K) = 0$.



*Eugenio Calabi, 1923-2023*

**EXERCISE:** Prove that **the form $\omega_J + \sqrt{-1}\,\omega_K$ is holomorphically symplectic on $(M, I)$.**

## Hyperkähler structure on a complex surface

**EXERCISE:** Let $M$ be a K3 surface, and $\omega$ a Kähler form. As in Lecture 7, define the decomposition $\Lambda^2 M = \Lambda^+ M \oplus \Lambda^- M$, where $\Lambda^\pm M$ denotes the $\pm$-eigenspaces of the Hodge $*$ operator acting on $\Lambda^2 M$. **Prove that $\Lambda^+ M$ is generated by the Kähler form and the forms $\operatorname{Re}\Omega$, $\operatorname{Im}\Omega$,** where $\Omega$ is a holomorphic symplectic form.

**PROPOSITION:** Let $M$ be a K3 surface, and $g$ a Kähler metric. Then the following are equivalent: (a) $g$ **is hyperkähler** (b) $g$ **is Ricci-flat.** (c) The bundle $\Lambda^+ M$ **is trivialized by parallel sections.**

**Proof. Step 1:** (a) $\Rightarrow$ (c) is clear, because $\omega_I, \omega_J, \omega_K$ trivialize $\Lambda^+ M$.

**Step 2:** To obtain (c) $\Rightarrow$ (b), note that the projection to $(2,0)$-part is parallel, hence $\Lambda^+ M \otimes_{\mathbb{R}} \mathbb{C}$ is trivialized by parallel sections of type $(2,0)$, $(0,2)$ and $(1,1)$. However, **a parallel section of type $(2,0)$ is closed, hence holomorphic.** The implication (b) $\Rightarrow$ (c) is clear.

**Step 3:** It remains to deduce (a) from (c). Using the metric, we identify $\mathfrak{so}(TM)$ and $\Lambda^2 TM$. This identification takes the decomposition $\Lambda^2 TM = \Lambda^+ M \oplus \Lambda^- M$ to the Lie algebra decomposition $\mathfrak{so}(4) = \mathfrak{so}(3) \oplus \mathfrak{so}(3)$, where each $\mathfrak{so}(3)$ is induced by the left and the right action of the Lie algebra of imaginary quiaternions on $\mathbb{H}$. **Taking an appropriate basis in $\mathfrak{so}(3)$, identified with the algebra of parallel sections of $\Lambda^+ M$, we obtain a parallel $\mathbb{H}$-action on $TM$.** ■