K3 surfaces

lecture 21: the Kähler cone of a K3 surface

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Demailly-Păun theorem

THEOREM: (Demailly-Păun)
Let M be a compact Kähler manifold, and $K \subset H^{1,1}(M,\mathbb{R})$ a subset consisting of all classes η such that $\int_Z \eta^p > 0$ for any p-dimensional complex subvariety $Z \subset M$. **Then the Kähler cone of M is one of the connected components of K.**

The proof of this theorem **is based on Demailly's regularization of positive currents and Calabi-Yau theorem.**

Compare this with Nakai-Moishezon theorem (Lecture 9):

THEOREM: (Nakai-Moishezon)
Let X be a compact projective variety, and L a line bundle on X. Assume that for any subvariety $Z \subset X$, $\dim Z = d$, one has $\int_Z c_1(L)^d > 0$. **Then L is ample.**

Example 1: Let M be a Kähler complex surface without curves (for example, a general K3 surface or a general complex torus). Then $K = \{\omega \in H^{1,1}(M,\mathbb{R}) \ \ \int_M \omega \wedge \omega > 0\}$. By Hodge index theorem, the intersection form on $H^{1,1}_{\mathbb{R}}(M)$ has signature $(1, h^{1,1} - 1)$, hence the set K has 2 connected components **(prove this)**. One of these components, denoted below as $\operatorname{Pos}(M)$, is the Kähler cone of M by Demailly-Păun theorem.

Demailly-Păun for Kähler surfaces: Buchdahl-Lamari theorem

Exercise 1: (sometimes also called **the Hodge index theorem**)
Let (V, q) be a vector space equipped with a scalar product of signature $(1, n)$, and $V^+ := \{x \in V \ \ | \ \ q(x,x) > 0\}$. **Prove that the set V^+ has two connected components,** K_1 and K_2, and **for any $x \in K_1$ and any non-zero y in the closure $\overline{K}_1$, we have $q(x,y) > 0$.**

REMARK: For complex surfaces, Demailly-Păun theorem **was proven a few years earlier by Buchdahl and Lamari,** who also used Demailly's regularization of positive currents.

THEOREM: Let M be a Kähler surface, containing a complex curve with non-negative self-intersection. Then its Kähler cone **is the set of all $\omega \in H^{1,1}(M, \mathbb{R})$ such that $\int_M \omega \wedge \omega > 0$ and $\int_C \omega > 0$ for all curves $C \subset M$.**

Proof: Next slide

Demailly-Păun for Kähler surfaces: Buchdahl-Lamari theorem (2)

THEOREM: Let M be a Kähler surface, containing a complex curve with non-negative self-intersection. Then its Kähler cone **is the set of all** $\omega \in H^{1,1}(M,\mathbb{R})$ **such that** $\int_C \omega > 0$ **for all curves** $C \subset M$ **and** $\int_M \omega \wedge \omega > 0$**.**

Proof. Step 1: This statement would follow from Demailly-Păun **if we prove that the set** K_1 $\omega \in H^{1,1}(M,\mathbb{R})$ **such that** $\int_C \omega > 0$ **for all curves** $C \subset M$ **and** $\int_M \omega \wedge \omega > 0$ **is connected.**

Step 2: The set

$$\{\omega \in H^{1,1}_{\mathbb{R}}(M) \ \ | \ \ \int_C \omega > 0 \ \text{ for all curves } C \subset M\}$$

is convex, and therefore connected. Let S be a curve with non-negative self-intersection. From Exercise 1, we obtain that $\int_S \omega > 0$ for all ω in one of connected components of the set $\{\omega \in H^{1,1}(M,\mathbb{R}) \ \ | \ \ \int_M \omega \wedge \omega > 0\}$ and $\int_S \omega < 0$ for all ω in the second component. Since both of these components are convex **(prove this)**, **the set** K_1 **is convex, being an intersection of two convex sets, and therefore connected.** ■

Demailly-Păun for K3 surfaces

THEOREM: Let M be a K3 surface, and $\mathfrak{R}$ the set of all (-2)-classes. Denote by $\operatorname{Pos}(M)$ the component of $\{\omega \in H^{1,1}(M,\mathbb{R}) \ \ | \ \ \int_M \omega\wedge\omega > 0\}$ which contains Kähler classes (by Exercise 1, there is one and only one such component). **Then the Kähler cone $\operatorname{Kah}(M)$ is a connected component of the set $\operatorname{Pos}(M)\backslash \bigcup_{r\in\mathfrak{R}} r^\perp$.**

Proof. Step 1: Let $C \subset M$ be a connected curve. Then either $(C,C) \geqslant 0$, and in this case $\int_C \eta > 0$ for any $\eta \in \operatorname{Pos}(M)$ (Exercise 1) or $(C,C) = -2$ (Lecture 19). **From Demailly-Păun it follows that**

$$\operatorname{Kah}(M) = \left\{\omega \in \operatorname{Pos}(M) \ \ | \ \ \int_Z \omega > 0 \ \text{ for all effective -2-classes } Z \in H_{1,1}(M,\mathbb{Z})\right\}.$$

Step 2: For any (-2)-class $z \in H^{1,1}(M,\mathbb{Z})$, either z or $-z$ is effective (Lecture 19). Let $\mathfrak{S}$ be the set of all subsets $\mathcal{S} \subset \mathfrak{R}$ such that for any $z \in \mathfrak{R}$ exactly one of z or $-z$ belong to $\mathcal{S}$. The set $E_M \subset H^{1,1}(M,\mathbb{Z})$ of effective (-2)-classes on M **is clearly an element of $\mathfrak{S}$.**

Step 3: Any $\mathcal{S} \in \mathfrak{S}$ defines a connected component $K_\mathcal{S}$ (possibly empty) of $W := \operatorname{Pos}(M)\backslash \bigcup_{r\in\mathfrak{R}} r^\perp$ via

$$K_\mathcal{S} := \{\eta \in \operatorname{Pos}(M) \ \ | \ \ (\eta, z) > 0 \quad \forall z \in \mathcal{S}\}.$$

Conversely, any non-empty connected component K of W defines a unique $\mathcal{S} \in \mathfrak{S}$ such that $K = K_\mathcal{S}$. From Step 1 we obtain that **the Kähler cone is equal to K_{E_M}.** ■

Demailly-Păun for K3 surfaces (2)

THEOREM: Let M be a K3 surface, and $\mathfrak{R}$ the set of all (-2)-classes. Denote by $\operatorname{Pos}(M)$ the component of $\{\omega \in H^{1,1}(M,\mathbb{R}) \quad \int_M \omega \wedge \omega > 0\}$ which contains Kähler classes (by Exercise 1, there is one and only one such component). **Then the Kähler cone $\operatorname{Kah}(M)$ is a connected component of the set $\operatorname{Pos}(M)\backslash \bigcup_{r\in\mathfrak{R}} r^\perp$.**

REMARK: Later we will prove that the mapping class group $\operatorname{Diff}(M)\operatorname{Diff}_0(M)$ **acts on the set of all components of $\operatorname{Pos}(M)\backslash \bigcup_{r\in\mathfrak{R}} r^\perp$ transitively,** hence **all of these connected components are Kähler cones of appropriate complex structures on a K3**.