K3 surfaces

lecture 22: the hyperkähler period space

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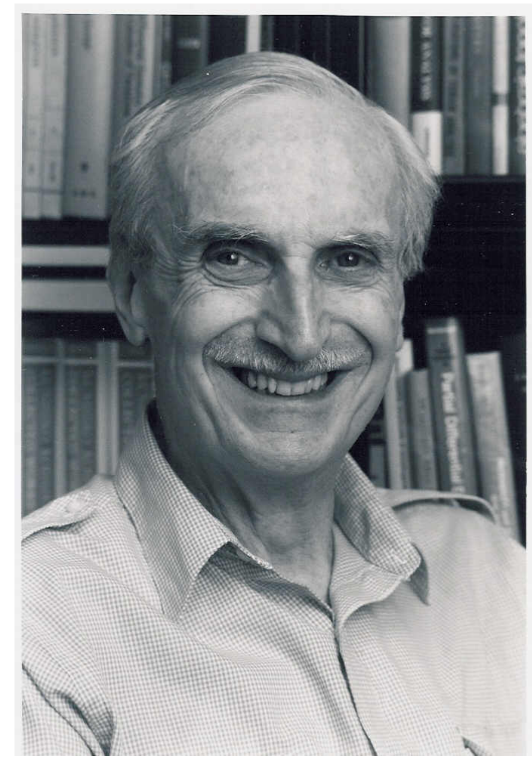
Hyperkähler manifolds: reminder

DEFINITION: (E. Calabi, 1978)
Let (M,g) be a Riemannian manifold equipped with three complex structure operators $I,J,K : TM \longrightarrow TM$, satisfying the quaternionic relations

$$I^2 = J^2 = K^2 = IJK = -\operatorname{Id}.$$

Suppose that g is Kähler with respect to I, J, K. Then (M,I,J,K,g) is called **hyperkähler**.

REMARK: This is the same as $\mathcal{H}ol(M) \subset Sp(n)$.
Indeed, if $\mathcal{H}ol(M) \subset Sp(n)$, we have 3 complex structures $I,J,K : TM \longrightarrow TM$, such that $\nabla(I) = \nabla(J) = \nabla(K) = 0$, which implies that I,J,K are Kähler. Conversely, if I,J,K are Kähler, we have $\nabla(I) = \nabla(J) = \nabla(K) = 0$.



Eugenio Calabi, 1923-2023

EXERCISE: Prove that **the form $\omega_J + \sqrt{-1}\,\omega_K$ is holomorphically symplectic on (M,I).**

Calabi-Yau theorem: reminder

DEFINITION: Let (M, I, ω) be a Kähler manifold. Its **Kähler class** is the cohomology class of ω in $H^{1,1}_{\mathbb{R}}(M, I)$.

DEFINITION: Let (M, I, ω) be a Kähler manifold with trivial canonical bundle, and $\Omega \in \Lambda^{n,0}_I(M)$ its non-degenerate holomorphic section. We say that (M, I) is **Ricci-flat** if $|\Omega| = const$, where $|\cdot|$ denotes the metric on $\Lambda^{n,0}_I(M)$ induced by the Kähler metric.

THEOREM: (Calabi-Yau)
Let (M, Ω) be compact holomorphically symplectic manifold of Kähler type, and $[\omega] \in H^{1,1}_{\mathbb{R}}(M, I)$ a Káhler class. **Then there exists a unique Ricci-flat metric g on (M, I) such that its Kähler class is equal to $[\omega]$.**

THEOREM: (Calabi)
A Kähler metric on a compact holomorphically symplectic manifold **is hyperkähler if and only if it is Ricci-flat.**

COROLLARY: A compact holomorphically symplectic manifold of Kähler type **admits a unique hyperkähler metric in each Kähler class.**

The twistor deformation

DEFINITION: Induced complex structures on a hyperkähler manifold are complex structures of form $S^2 \cong \{L := aI + bJ + cK, \ \ | \ \ a^2 + b^2 + c^2 = 1\}$. **They are usually non-algebraic**. Indeed, if M is compact, then, for generic a, b, c, the manifold (M, L) **has no divisors (Fujiki)**.

DEFINITION: A **twistor space** $\operatorname{Tw}(M)$ of a hyperkähler manifold is **a complex manifold obtained by gluing these complex structures into a holomorphic family over $\mathbb{C}P^1$.** More formally:

Let $\operatorname{Tw}(M) := M \times S^2$. Consider the complex structure $I_m : T_m M \to T_m M$ on M induced by $J \in S^2 \subset \mathbb{H}$. Let I_J denote the complex structure on $S^2 = \mathbb{C}P^1$.

The operator $I_{\operatorname{Tw}} = I_m \oplus I_J : T_x \operatorname{Tw}(M) \to T_x \operatorname{Tw}(M)$ satisfies $I_{\operatorname{Tw}}^= - \operatorname{Id}$. **It defines an almost complex structure on $\operatorname{Tw}(M)$.** This almost complex structure is known to be integrable (Obata, Salamon)

EXAMPLE: If $M = \mathbb{H}^n$, $\operatorname{Tw}(M) = \operatorname{Tot}(\mathcal{O}(1)^{\oplus n}) \cong \mathbb{C}P^{2n+1} \backslash \mathbb{C}P^{2n-1}$

REMARK: If M is compact, $\operatorname{Tw}(M)$ never admits a Kähler structure.

Gromov-Hausdorff metrics

DEFINITION: Let $X \subset M$ be a subset of a metric space, and $y \in M$ a point. **Distance from a y to X** is $\inf_{x\in X} d(x,y)$. **Hausdorff distance** $d_H(X,Y)$ between to subsets $X, Y \subset M$ of a metric space is

$$\max(\sup_{x\in X} d(x,Y), \sup_{y\in Y} d(y,X)).$$

Gromov-Hausdorff distance between complete metric spaces X, Y of diameter $\leqslant d$ is an infimum of $d_H(\varphi(X), \psi(Y))$ taken over all isometric embeddings $\varphi: X \longrightarrow Z$, $\psi: Y \longrightarrow Z$ to a third metric space.

REMARK: This definition **puts a structure of a metric space** on the set of equivalence classes of all complete metric spaces with diameter $\leqslant d$.

REMARK: Let $\varphi: X \longrightarrow Y$ be a map of metric spaces (not necessarily continuous). Its **defect** δ_φ is $\inf_{x_1,x_2\in X} |d(x_1,x_2) - d(\varphi(x_1),\varphi(x_2))|$. Gromov-Hausdorff distance between metric spaces X, Y is bounded (both directions) by the quantity $\hat{d}_{GH}(X,Y) = \inf_{\varphi,\psi} \max(\delta_{\psi\circ\varphi}, \delta_{\varphi\circ\psi})$, where infimum is taken over the set of all maps $\varphi: X \longrightarrow Y$, $\psi: Y \longrightarrow X$:

$$C_1 \hat{d}_{GH}(X,Y) \leqslant d_{GH}(X,Y) \leqslant C_2 \hat{d}_{GH}(X,Y)$$

This implies, in particular, that **a converging sequence of Riemannian metrics converges in Gromov-Hausdorff topology.**

The hyperkähler Teichmüller space (reminder)

DEFINITION: Let Teich_h be the Teichmüller space of conformal classes of hyperkähler metrics on a K3 surface M, that is, the space of all hyperkähler metrics on M up to $\mathbb{R}^{>0} \times \operatorname{Diff}_0$-action. We will call Teich_h **"the hyperkähler Teichmüller space"**.

REMARK: It is not hard to see that the topology on Teich_h, identified with the set of hyperkähler metrics of diameter 1, is induced by d_{GH}, **and therefore Teich_h is Hausdorff.**

Oswald Teichmüller, 1913-1943

The hyperkähler period map

REMARK: Let M be a compact Riemannian manifold, and $\mathcal{H}^2(M,\mathbb{R})$ be the space of harmonic 2-forms, tacitly identified with the second cohomology. Since the Hodge star operator commutes with the Laplacian, it preserves $\mathcal{H}^2(M,\mathbb{R})$. Denote by $H^+(M,\mathbb{R})$ the set of $*$-invariant harmonic forms, and $H^-(M,\mathbb{R})$ the set of $*$-anti-invariant harmonic forms; then $H^2(M) = H^+(M) \oplus H^-(M)$, and the intersection form is positive on $H^+(M)$ and negative on $H^-(M)$. This decomposition **defines a map from the space of all Riemannian metrics to the Grassmanian of all $\dim H^+(M)$-dimensional positive subspaces in $H^2(M,\mathbb{R})$.**

REMARK: To simplify the language, in the sequel **we will identify hyperkähler structures (I,J,K,g) and $(I,J,K,\lambda g)$, where λ is a constant**. For a K3, **the triple (I,J,K) determines g uniquely up to a constant**, hence we will say "hyperkähler structure I,J,K" signifying "(I,J,K,g) up to a constant multiplier".

DEFINITION: Given a hyperkähler structure I,J,K on a K3 surface, consider the decomposition $\Lambda^2(M) = \Lambda^+(M) \oplus \Lambda^-(M)$. The bundle $\Lambda^+(M)$ is trivialized by parallel sections $\omega_I, \omega_J, \omega_K$, which generate the subspace $H^+(M,\mathbb{R}) \subset \mathcal{H}^2(M,\mathbb{R})$. Let $\operatorname{Gr}_{+++}(H^2(M,\mathbb{R}))$, or simply Gr_{+++}, be the Grassmannian of 3-dimensional positive oriented subspaces in $H^2(M,\mathbb{R})$. Define **the hyperkähler period map** as the map $\operatorname{Per}_h : \operatorname{Teich}_h \longrightarrow \operatorname{Gr}_{+++}(H^2(M,\mathbb{R}))$ taking I,J,K to the corresponding space $H^+(M) = \langle \omega_I, \omega_J, \omega_K \rangle$.

The hyperkähler period space

DEFINITION: Let $\mathbb{P}\mathrm{er}_h \subset \mathrm{Gr}_{+++}$ be the set of all $W \in \mathrm{Gr}_{+++}(H^2(M,\mathbb{R}))$ such that $W^\perp$ does not contain (-2)-classes. We call $\mathbb{P}\mathrm{er}_h$ **the hyperkähler period space** of a K3 surface M.

PROPOSITION: Let M be a K3 surface. Consider the hyperkähler period map $\mathrm{Per}_h : \mathrm{Teich}_h \longrightarrow \mathrm{Gr}_{+++}$. **Then $\mathrm{Per}_h(g) \in \mathbb{P}\mathrm{er}_h$ for any hyperkähler structure g.**

Proof. Step 1: Let $W \subset \Lambda^2(M,\mathbb{R})$ be the space generated by $\omega_I, \omega_J, \omega_K$ for a hyperkähler structure g. The corresponding twistor deformation is parametrized by all quaternions $L = aI + bJ + cK$, $L^2 = -1$, and any 2-dimensional oriented plane $V \subset W$ is obtained as $\langle \mathrm{Re}\,\Omega, \mathrm{Im}\,\Omega\rangle$ for an appropriate C-symplectic form $\Omega \in W \otimes_{\mathbb{R}} \mathbb{C}$.

Step 2: Assuming the contrary, let $W = \mathrm{Per}_h(g)$ be a space such that $W^\perp$ contains a (-2)-class η, and I a complex structure induced by a C-symplectic form $\Omega \in W \otimes_{\mathbb{R}} \mathbb{C}$. Since η is orthogonal to $\langle \mathrm{Re}\,\Omega, \mathrm{Im}\,\Omega\rangle$, it has Hodge type (1,1) on (M,I). Since $\eta \in W^\perp$, we have $\int_M \omega_I \wedge \eta = 0$, where $\omega_I \in W$ is the Kähler form of (M,I). **This is impossible, because either η or $-\eta$ is effective, and $\int \omega_I \wedge \eta > 0$ for all effective classes.** ■

Hyperkähler local Torelli theorem

PROPOSITION: The hyperkähler period map $\operatorname{Per}_h : \operatorname{Teich}_h \longrightarrow \mathbb{P}\mathrm{er}_h$ **is locally a homeomorphism.**

Proof. Step 1: Let Teich'_h be the space of triples $\omega_I, \omega_J, \omega_K$, and $\mathbb{P}\mathrm{er}'_h$ the space of pairwise orthogonal classes $a \bot b \bot c \in H^2(M, \mathbb{R})$ satisfying $\int_M a \wedge a = \int_M b \wedge b = \int_M c \wedge c > 0$. Since Teich'_h is $SO(3) \times \mathbb{R}^{>0}$-fibered over Teich_h and $\mathbb{P}\mathrm{er}'_h$ is $SO(3) \times \mathbb{R}^{>0}$-fibered over $\mathbb{P}\mathrm{er}_h$, **it suffices to prove that the natural period map $\operatorname{Per}'_h : \operatorname{Teich}'_h \longrightarrow \mathbb{P}\mathrm{er}'_h$ is locally a diffeomorphism.**

Step 2: The map $\operatorname{Per}'_h : \operatorname{Teich}'_h \longrightarrow \mathbb{P}\mathrm{er}'_h$ takes a triple $(\omega_I, \omega_J, \omega_K)$ to the cohomology classes $([\omega_I], [\omega_J], [\omega_K])$. By local Torelli for C-symplectic structures, **the map taking the pair (ω_J, ω_K) to $([\omega_J], [\omega_K])$ is locally a homeomorphism.** The forgetful arrow $\operatorname{Teich}'_h \longrightarrow \mathbb{P}\mathrm{er}_C$ taking $\omega_I, \omega_J, \omega_K$ to the C-symplectic structure $\Omega = \omega_J + \sqrt{-1}\,\omega_K$ **has fiber $\operatorname{Kah}(M, \Omega)$ by Calabi-Yau theorem,** hence it is a smooth submersion with the fiber locally homeomorphic to $H^{1,1}_{\mathbb{R}}(M, \Omega)$. The forgetful map $\mathbb{P}\mathrm{er}'_h \longrightarrow H^2(M, \mathbb{C})$ taking $([\omega_I], [\omega_J], [\omega_K])$ to (ω_J, ω_K) is also a smooth submersion, with the fiber locally homeomorphic to $H^{1,1}_{\mathbb{R}}(M, \Omega) = \langle [\omega_J], [\omega_K] \rangle^{\bot}$.

Hyperkähler local Torelli theorem

PROPOSITION: The The hyperkähler period map $\operatorname{Per}_h : \operatorname{Teich}_h \longrightarrow \mathbb{P}\mathrm{er}_h$ **is locally a homeomorphism.**

Step 2: The map $\operatorname{Per}'_h : \operatorname{Teich}'_h \longrightarrow \mathbb{P}\mathrm{er}'_h$ takes a triple $(\omega_I, \omega_J, \omega_K)$ to the cohomology classes $([\omega_I], [\omega_J], [\omega_K])$. By local Torelli for C-symplectic structures, **the map taking the pair (ω_J, ω_K) to $([\omega_J], [\omega_K])$ is locally a homeomorphism.** The forgetful arrow $\operatorname{Teich}'_h \longrightarrow \mathbb{P}\mathrm{er}_C$ taking $\omega_I, \omega_J, \omega_K$ to the C-symplectic structure $\Omega = \omega_J + \sqrt{-1}\,\omega_K$ **has fiber $\operatorname{Kah}(M, \Omega)$ by Calabi-Yau theorem,** hence it is a smooth submersion with the fiber locally homeomorphic to $H^{1,1}_{\mathbb{R}}(M, \Omega)$. The forgetful map $\mathbb{P}\mathrm{er}'_h \longrightarrow H^2(M, \mathbb{C})$ taking $([\omega_I], [\omega_J], [\omega_K])$ to (ω_J, ω_K) is also a smooth submersion, with the fiber locally homeomorphic to $H^{1,1}_{\mathbb{R}}(M, \Omega) = \langle [\omega_J], [\omega_K] \rangle^{\perp}$.

Step 3: Consider the diagram

$$\begin{array}{ccc} \operatorname{Teich}'_h & \xrightarrow{\operatorname{Per}'_h} & \mathbb{P}\mathrm{er}'_h \\ \downarrow & & \downarrow \\ \mathbb{P}\mathrm{er}_C & \xrightarrow{\operatorname{Per}_C} & H^2(M, \mathbb{C}). \end{array}$$

Its bottom arrow is locally a homeomorphism, and the vertical arrows are fibrations with the fibers with are locally identified by the top arrow Per'_h. **This implies that the top arrow is also locally a homeomorphism.** ■

The hyperkähler global Torelli theorem

In the next lecture, I will prove the global Torelli theorem

THEOREM: The hyperkähler period map $\operatorname{Per}_h : \operatorname{Teich}_h \longrightarrow \mathbb{P}\mathrm{er}_h$ **is a homeomorphism** for any connected component of Teich_h.

REMARK: Conjecturally, Teich_h is connected. **This is equivalent to the natural map**

$$\frac{\operatorname{Diff}(M)}{\operatorname{Diff}_0(M)} \longrightarrow O^+(H^2(M,\mathbb{Z}))$$

being an isomorphism; here $O^+(H^2(M,\mathbb{Z}))$ is an index 2 subgroup of $O(H^2(M,\mathbb{Z}))$ generated by reflections in (-2)-classes.

REMARK: $O^+(H^2(M,\mathbb{Z}))$ is an index 2 subgroup in $O(H^2(M,\mathbb{Z}))$ by results of C. T. C. Wall: *C. T. C. Wall, On the orthogonal groups of unimodular quadratic forms II, J. Reine Angew. Math. 213. (1963/64), 122-136.* The group $O^+(H^2(M,\mathbb{Z}))$ is the image of the mapping class group acting on cohomology by results of S. Donaldson and C. Borcea,

[B] *C. Borcea, Diffeomorphisms of a K3 surface, Math. Ann. 275 (1986),*

[D] *S. K. Donaldson, The orientation of Yang-Mills moduli spaces and 4-manifold topology, J. Diff. Geom. 26 (1987), 397-428.*

[BL] *Benson Farb, Eduard Looijenga, Moduli spaces and period mappings of genus one fibered K3 surfaces,* `https://arxiv.org/abs/2303.08214`