K3 surfaces

lecture 23: Metric structures associated with twistor families

Misha Verbitsky

IMPA, sala 236

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Sub-Riemannian structures

DEFINITION: Let M be a Riemannian manifold and $B \subset TM$ a sub-bundle. A **horizontal path** is a piecewise smooth path $\gamma: [b,a] \longrightarrow M$ tangent to B everywhere. A **sub-Riemannian**, or **Carno-Carathéodory** metric M is

$$d_B(x,y) := \inf_{\gamma \text{ horizontal}} L(\gamma):$$

the infimum of the length $L(\gamma)$ for all horizontal paths connecting x to y.

DEFINITION: Let $B \subset TM$ be a sub-bundle, and $B = B_0 \subset B_1 \subset ...$ a sequence of subsheaves defined by $[B_i, B_i] = B_{i+1}$. We say that $B \subset M$ **satisfies the Chow-Rashevskii condition** if $B_s = TM$ for s sufficiently big.

THEOREM: (Chow-Rashevskii theorem; 1938, 1939) Let $B \subset M$ be a sub-bundle which satisfies the Chow-Rashevskii condition. **Then any two points can be connected by a horizontal path,** and the sub-Riemannian metric d_B is finite. Moreover, **the corresponding topology on M is equivalent to the usual topology.**

REMARK: Subriemannian matric is an example of **intrinsic metric**, or **path metric**. I will now define this notion in full generality, following http://verbit.ru/IMPA/METRIC-2023/ and *Burago, D., Burago, Y., and Ivanov, S., A course in metric geometry, AMS Graduate Studies in Mathematics Volume 33 (2001).* Full proofs, exercises and more examples are found either of these sources.

Arc-length of a path

Let (M, d) be a metric space, and $\gamma: [a,b] \mapsto M$ a continuous path (here $[a,b]$ denotes the closed interval). Let $x_0 = a < x_1 < ... < x_{n-1} < b = x_n$ be the partition of the interval, and $L_\gamma(x_1, ...x_{n-1}) := \sum_{i=0}^{n-1} d(\gamma(x_i), \gamma(x_{i+1}))$ the length of the corresponding chain.

DEFINITION: We define **the arc-length** (or **the length**) of the path γ as

$$L_d(\gamma) := \sup_{a<x_1<...<x_{n-1}<b} L_\gamma(x_1, ...x_{n-1}),$$

where supremum is taken over all partitions of the interval $[a,b]$. A path is called **rectifiable** if its arc-length is finite.

EXERCISE: Prove the following properties of arc-length.

* **The arc-length is additive:** for any path $\gamma: [a,c] \longrightarrow M$ and any $b \in [a,c]$, one has $L_d(\gamma) = L_d(\gamma|_{[a,b]}) + L_d(\gamma|_{[b,c]})$.
* **The arc-length is continuous as a function of the ends:** for any rectifiable path $\gamma: [a,c] \longrightarrow M$, **the function $L(\gamma\Big|_{[a,b]})$ depends on $b \in [a,c]$ continuously.**
* **The arc-length is compatible with the metric:** for any $x, y \in M$, and any path $\gamma: [a,b] \mapsto M$ with ends in x, y, we have $L_d(\gamma) \geqslant d(x,y)$.
* **The arc-length is invariant under the reparametrizations:** for any homeomorphism $\varphi: [a,b] \longrightarrow [a,b]$ the arc-length of γ is equal to the arc-length of $\varphi \circ \gamma$.

Intrinsic metric

THEOREM: Let $\gamma : [a,b] \longrightarrow \mathbb{R}^n$ be a smooth path in $\mathbb{R}^n$ with the standard metric. **Then $L_d(\gamma) = \int_a^b |\gamma'(t)| dt$.**

DEFINITION: A metric d on M us called **intrinsic metric**, or **path metric**, if $d(x,y) = \inf_\gamma L_d(\gamma)$, where the infimum is taken over all rectifiable paths γ connecting x to y.

DEFINITION: Let M be a topological space. **A class of admissible paths** is a set $\mathcal{C}$ of continuous maps $[a,b] \longrightarrow M$ with the following properties.

1. **The concatenation.** For any two paths $[a,b] \xrightarrow{\gamma_1} M$, $[b,c] \xrightarrow{\gamma_2} M$, satisfying $\gamma_1(b) = \gamma_2(b)$, **the concatenation** $\gamma : [a,c] \longrightarrow M$ (that is, the path equal to γ_1 on $[a,b]$ and to γ_2 on $[b,c]$) is also admissible.

2. **Linear reparametrization.** For any linear map $\varphi : [a,b] \longrightarrow [c,d]$ and any admissible path $\gamma : [c,d] \longrightarrow M$, the path $\varphi \circ \gamma$ is also admissible.

3. **Restriction.** For each path $[a,b] \xrightarrow{\gamma} M$, and an interval $[c,d] \subset [a,b]$, the restriction $\gamma|_{[c,d]}$ is also admissible.

Length structures

DEFINITION: Let M be a topological space. **A length structure** on M is a class $\mathcal{C}$ of admissible paths together with **a length functional** $L: \mathcal{C} \longrightarrow \mathbb{R}^{\geqslant 0} \cup \infty$ which satisfies the following axioms.

1. **Additivity:** for any path $\gamma: [a,c] \longrightarrow M$ and any $b \in [a,c]$, one has $L(\gamma) = L(\gamma|_{[a,b]}) + L(\gamma|_{[b,c]})$.

2. **The length is continuous as a function of the ends:** for any path $\gamma: [a,c] \longrightarrow M$, **the function $L(\gamma\Big|_{[a,b]})$ depends on $b \in [a,c]$ continuously.**

3 **Invariance under reparametrizations:** for any homeomorphism $\varphi: [a,b] \longrightarrow [a,b]$ and any admissible path $\gamma: [a,c] \longrightarrow M$ such that $\varphi \circ \gamma$ is also admissible, one has $L(\gamma) = L(\varphi \circ \gamma)$.

4. **Compatibility with the topology:** for any point $x \in M$, and any closed set $Z \subset M$, such that $x \notin Z$, there is a number $\varepsilon > 0$ such that $L(\gamma) > \varepsilon$ for any admissible path connecting Z to x.

EXAMPLE: The arc-length on the class of rectifiable paths **gives a length structure.**

The metric associated with a length structure

DEFINITION: Let M be a topological space equipped with a length structure L. The **path metric** d_L associated with L is defined as $d_L(x, y) := \inf_\gamma L(\gamma)$, where the infimum is taken over all admissible paths connecting x tp y.

CLAIM: **d_L is an intrinsic metric, and all intrinsic metrics are obtained this way.**

EXAMPLE: Let $M = \mathbb{R}^n$ with the standard topology, and $\mathcal{C}$ the class of all piecewise-linear paths (polygonal chains) $[x_0, x_1] \cup [x_1, x_2] \cup ... \cup [x_{n-1}, x_n]$. Define the length functional taking $\gamma = [x_0, x_1] \cup [x_1, x_2] \cup ... \cup [x_{n-1}, x_n] \in \mathcal{C}$ to $L(\gamma) = \sum |d(x_i, x_{i+1})|$.

CLAIM: The metric d_L constructed from this length structure **is equal to the standard Euclidean metric.**

Finsler metrics

EXAMPLE: (**A Finsler metric**)
Let $\nu_x : T_x X \longrightarrow \mathbb{R}$ the norm on the tangent space, continuously depending on x. For any piecewisely smooth path $\gamma : [a,b] \longrightarrow X$, define $L_\nu(\gamma(t)) := \int_a^b \nu_{\gamma(t)}(\gamma'(t))dt$.

PROPOSITION: This defines a length functional on the class of piecewisely smooth paths.

DEFINITION: The corresponding metric on X is called **a Finsler metric**.

DEFINITION: Suppose now that ν_x is defined on a sub-bundle $B \subset TX$ satisfying the Chow-Rashevskii condition. As above, define the same length functional $L_\nu(\gamma(t))$ on the class of all piecewise smooth paths tangent to B. The corresponding metric on X is called **a sub-Finsler metric**.

Pencils of 3-spaces

DEFINITION: Let (H, q) be a quadratic vector space of signature $(3, n)$, and $W_1, W_2 \in \mathrm{Gr}_{+++}$ 3-spaces such that $\dim W_1 \cap W_2 = 3$. The space $U = W_1 + W_2$ is 4-dimensional, because $W_1 \cap W_2$ is 2-dimensional. All 3-spaces in U are lines in U^*, that is, points in $\mathbb{P}U^*$; this equivalence can be obtained by associating with W the set $\mathrm{Ann}(W)$ of all $\lambda \in U^*$ vanishing on W. The points $W_1, W_2 \in \mathbb{P}U^*$ are connected by a unique line $\mathbb{P}l \subset \mathbb{P}U^*$, where $l \subset U^*$ is a 2-dimensional subspace obtained as by $l = \mathrm{Ann}(W_1) + \mathrm{Ann}(W_2)$. The corresponding family of 3-spaces is called **a 1-dimensional pencil of 3-spaces in U**.

REMARK: The space U^* inherits a natural quadratic form q from U, which has signature $(3, 1)$ **(prove this).**

Positive segments connecting positive 3-spaces

CLAIM: The quadratic form q restricted to $l \subset U^*$ has signature (1,1).

Proof: A point $u \in U^*$ corresponds to a positive 3-space when $q(u,u) > 0$ and to a signature (2,1)-space otherwise **(prove it).** Since $W_1 + W_2$ has signature $(3,1)$, and W can be obtained as union of all W_t, $t \in l$ **(prove it)**, there are both positive and mixed signature points in $\mathbb{P}l$. ■

REMARK: The space $\mathbb{P}l = \mathbb{R}P^1$ is a circle. The 2-plane l is split onto 4 quadrants by two lines in $\{x \in l \mid q(x,x) = 0\}$, hence $\mathbb{P}l = \mathbb{R}P^1$ **has only two quadrants, positive and negative.** In other words, as u rotates along the circle $\mathbb{P}l$, the value of $q(u,u)$ **changes sign twice.**

DEFINITION: The **positive segment** connecting W_1 and W_2 is the only segment in $\mathbb{P}l = \mathbb{R}P^1$ connecting W_1 to W_2 and consisting of $u \in \mathbb{P}l$ with $q(u,u) > 0$.

Subtwistor chains

DEFINITION: Let Gr_{+++} be the Grassmannian of oriented 3-spaces in $H^2(M,\mathbb{R}) = \mathbb{R}^{3,19}$, where M is a K3 surface, and $\mathrm{Per}_h \subset \mathrm{Gr}_{+++}$ be the hyperkähler period space, that is, the space of all $W \in \mathrm{Gr}_{+++}$ such that $W^\perp$ does not contain (-2)-vectors (integer classes η such that $\int_M \eta \wedge \eta = -2$). A **subtwistor segment** is a positive segment in Per_h connecting 3-spaces W_1, W_2 such that $W_1 \cap W_2$ is a 2-plane V such that $V^\perp$ does not contain (-2)-vectors.

DEFINITION: Let $\mathrm{Per}_h \subset \mathrm{Gr}_{+++}$ be the hyperkähler period space, and $\gamma := I_1 \cup I_2 \cup, ..., I_n$ be a sequence of subtwistor segments such that the end of I_k is equal to the beginning of I_{k+1}. Then I is called **a subtwistor chain**.

Subtwistor metric on the hyperkähler period space

REMARK: Recall that $SO^+(H^2(M,\mathbb{R}))$ **denotes the connected component of unity in $SO(H^2(M,\mathbb{R}))$.**

EXERCISE: Consider the space $\operatorname{Gr}_{+++} = \frac{SO^+(H^2(M,\mathbb{R}))}{SO(3)\times SO(19)}$. **Prove that Gr_{+++} admits a unique, up to a constant, $SO^+(H^2(M,\mathbb{R}))$-invaiant Riemannian metric.**

REMARK: This metric can be defined explicitly as follows. Let $W \in \operatorname{Gr}_{+++}$. Then $T_W \operatorname{Gr}_{+++} = \operatorname{Hom}(W, W^\perp)$. The space $\operatorname{Hom}(W, W^\perp) = W^* \otimes_{\mathbb{R}} W^\perp$ is a tensor product of a vector space with positive definite Euclidean metric and a space with negative definite Euclidean metric; flipping the sign, we obtain a product of two spaces with Euclidean metric, and **a tensor product of two Euclidean spaces is Euclidean.**

REMARK: From now on, **we will treat Gr_{+++} as a Riemannian manifold,** with the Riemannian metric defined above.

DEFINITION: Let γ be a subtwistor chain. Define its length $L(\gamma)$ as the length of the corresponding path in Gr_{+++}. Define **the subtwistor metric** $d_{tw}(x,y)$ on Gr_{+++} as infimum of $L(\gamma)$, for all subtwistor chains connecting x to y.

Cone structures

DEFINITION: Let X be a locally G-homogeneous space, and $C \subset \operatorname{Tot}(TX)$ a G-invariant, $\mathbb{R}$-invariant subset. We say that C is **a cone structure** if C generates a sub-bundle $B \subset TX$ which satisfies the Chow-Rashevskii condition, and, moreover, for each $x \in M$ the intersection $C_x := C \cap T_x M$ is open and dense in its fiberwise closure $\overline{C}_x$, and the union of all $\overline{C}_x$ the closure of C.

DEFINITION: Fix a reference Riemannian metric g on X, and define **the cone metric** $d_C(x,y)$ as as infimum of the Riemannian length $L(\gamma)$, for all piecewise smooth paths connecting x to y which are tangent to C.

(D. Korshunov)
The cone metric is sub-Finsler.

REMARK: From Chow-Rashevskii theorem it follows also that **all sub-Finsler metrics define the standard topology on X.**

Cone structures and subtwistor metric

Let M be a K3, Gr_{+++} the corresponding Grassmann space, and $C \subset T\operatorname{Gr}_{+++}$ the set of all vectors tangent to subtwistor segments.

CLAIM: Let $W \in \operatorname{Gr}_{+++}$ be a point, and $C_W := T_W \operatorname{Gr}_{+++} \cap C$. Denote by $\overline{C}_W$ its closure. Then $\overline{C}_W \subset T_W \operatorname{Gr}_{+++} = \operatorname{Hom}(W, W^\perp)$ **is the set of all matrices of rank 1** in $\operatorname{Hom}(W, W^\perp)$. Moreover, **the union of all $\overline{C}_W$ is the closure of C.**

Proof. Step 1: A subtwistor segment corresponds to a family of 3-spaces which lies in a 4-dimensional subspace of $V = H^2(M, \mathbb{R})$. The corresponding matrix $A \in \operatorname{Hom}(W, W^\perp)$ has rank 1; conversely, if $A \in \operatorname{Hom}(W, W^\perp)$ has rank 1, the corresponding family of 3-spaces lies a 4-dimensional subspace of V generated by W and $A(W)$. Denote by $\underline{C} \subset \operatorname{Tot}(T\operatorname{Gr}_{+++})$ the set of all tangent vectors associated with matrices of rank 1. This set is clearly G-invariant, closed and generates $T\operatorname{Gr}_{+++}$. **It remains only to show that the union of all $\overline{C}_W$ is $\underline{C}$.**

Cone structures and subtwistor metric (2)

Let M be a K3, Gr_{+++} the corresponding Grassmann space, and $C \subset T\operatorname{Gr}_{+++}$ the set of all vectors tangent to subtwistor segments.

CLAIM: Let $W \in \operatorname{Gr}_{+++}$ be a point, and $C_W := T_W \operatorname{Gr}_{+++} \cap C$. Denote by $\overline{C}_W$ its closure. Then $\overline{C}_W \subset T_W \operatorname{Gr}_{+++} = \operatorname{Hom}(W, W^\perp)$ **is the set of all matrices of rank 1** in $\operatorname{Hom}(W, W^\perp)$. Moreover, **the union of all $\overline{C}_W$ is the closure of C.**

Proof. Step 1: It remains only to show that the union of all $\overline{C}_W$ is $\underline{C}$.

Step 2: Every subtwistor segment $\gamma: [0,1] \longrightarrow \operatorname{Gr}_{+++}$ tangent to $W \in \operatorname{Gr}_{+++}$ belongs to a 4-dimensional space W_1, with an extra condition that the orthogonal to the 2-dimensional space $V_t := W \cap \gamma(t)$ does not contain (-2)-classes. Let $A \in \operatorname{Hom}(W, W^\perp)$ be the tangent vector to γ. Then $V_t = \ker A$. The set of all 2-planes $V_1 \subset W$ such that $V_1^\perp$ contains (-2)-classes is countable **(prove this).** Therefore, **the set of all $A \in \operatorname{Hom}(W, W^\perp)$ which are tangent to subtwistor segments is dense in $\underline{C} \cap T_W \operatorname{Gr}_{+++}$.** ■

COROLLARY: Subtwistor metric on Gr_{+++} is Finsler.

Proof: Indeed, the subtwistor metric is a special case of cone metric, and it is sub-Finsler by Korshunov's theorem, and Finsler when the cone generates the tangent space. ■