K3 surfaces

lecture 24: Covering maps and the Torelli theorem for hyperkähler structures

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November 25, 2024, 17:00

Covering maps

DEFINITION: Let $\varphi: \tilde{M} \longrightarrow M$ be a continuous map of manifolds (or CW complexes). We say that φ is **a covering** if φ is locally a homeomorphism, and for any $x \in M$ there exists a neighbourhood $U \ni x$ such that is a disconnected union of several manifolds U_i such that the restriction $\varphi\Big|_{U_i}$ is a homeomorphism.

REMARK: From now on, we always assume that our topological spaces **are M locally contractible.**

THEOREM: A local homeomorphism of compacts spaces is a covering.

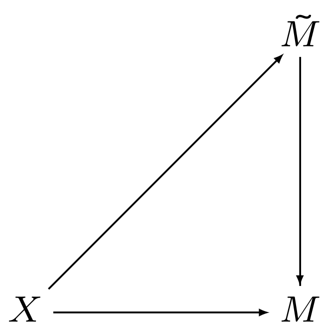
DEFINITION: Let Γ be a discrete group continuously acting on a topological space M. This action is called **properly discontinuous** if M is locally compact, and the space of orbits of Γ is Hausdorff.

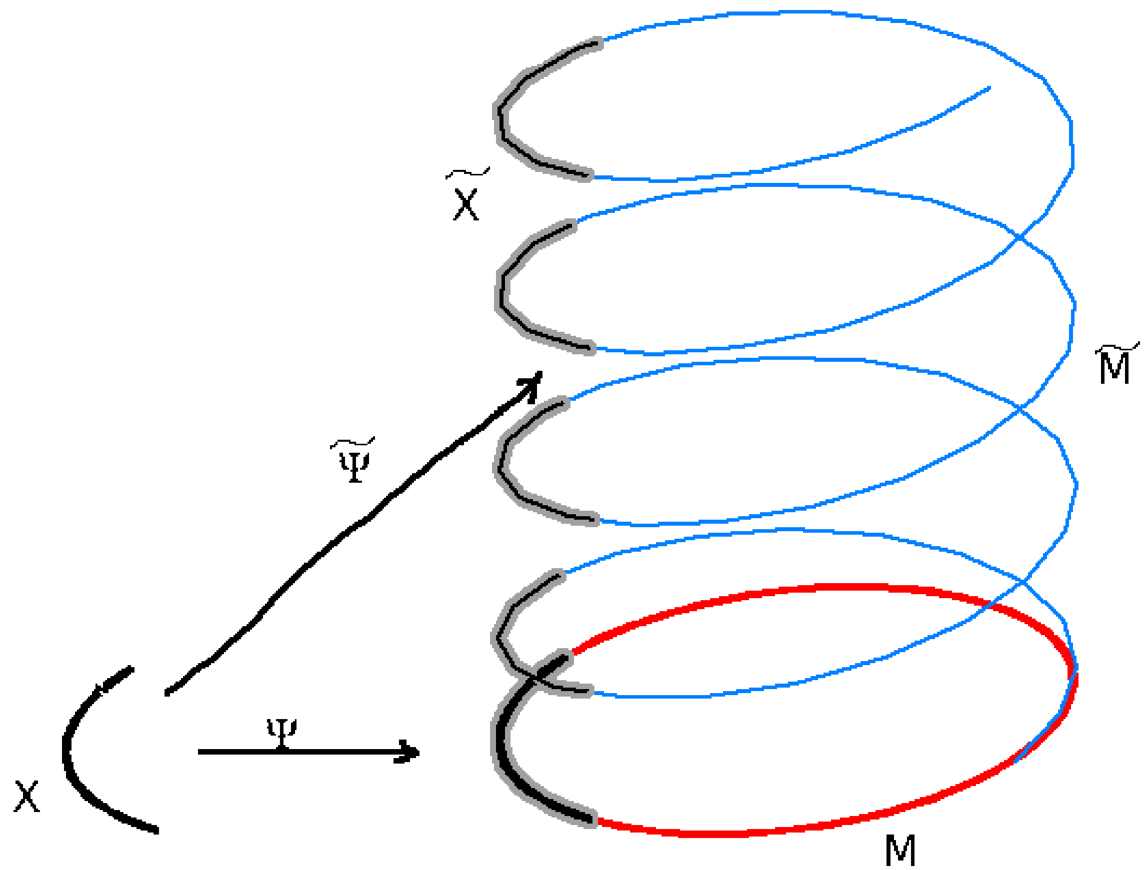
THEOREM: Let Γ be a discrete group acting on M properly discontinuously. Suppose that the stabilizer group $\Gamma' : \operatorname{St}_\Gamma(x)$ is the same for all $x \in M$. **Then $M \longrightarrow M/\Gamma$ is a covering.** Moreover, **all covering maps are obtained like that.**

These results are left as exercises.

Homotopy lifting principle

THEOREM: (homotopy lifting principle)
Let X be a simply connected topological space, and $\varphi : \tilde{M} \longrightarrow M$ a covering map. Fix $x \in X$. Then for each continuous map $X \longrightarrow M$, and each $\tilde{x} \in \varphi^{-1}(x)$ there exists a unique lifting $X \longrightarrow \tilde{M}$ passing through $\tilde{x}$ and making the following diagram commutative.





Path lifting for covering maps

DEFINITION: Let $\varphi: \tilde M \longrightarrow M$ be a continuous map. We say that φ is a **local homeomorphism** if for each $x\in \tilde M$ there exists an open neighbourhood $U\ni x$ such that $\varphi(U)$ is open and $\varphi: U \longrightarrow \varphi(U)$ is a homeomorphism. A path $\tilde\gamma: [a,b]\longrightarrow \tilde M$ is **a lifting of a path** $\gamma: [a,b]\longrightarrow M$ if $\gamma(t)=\varphi(\tilde\gamma(t))$ for all $t\in [a,b]$. We say that a path $\gamma: [a,b]\longrightarrow M$ starting in x **has lifting property** if for each $\tilde x\in \varphi^{-1}(x)$ there exists lifting $\tilde\gamma: [a,b]\longrightarrow \tilde M$ starting in $\tilde x$.
REMARK: For any local homeomorphism, the path lifting $\tilde\gamma$ **is uniquely determined by γ and $\tilde x$.**

REMARK: From homotopy lifting principle, it is clear that **coverings have path lifting property for all paths $\gamma: [a,b]\longrightarrow M$.**

The converse is also true:
CLAIM: Let $\varphi: \tilde M\longrightarrow M$ be a local homeomorphism of manifolds which has path lifting for all paths. **Then φ is a covering.**
Proof: Put a coordinate system on $U\subset M$, with center in x. Given a point $y\in U$, take a geodesic interval γ_y (with respect to flat metric on U) connecting y to x. For each $\tilde x\in \varphi^{-1} * x$), the path γ_y can be lifted to a path starting in $\tilde x$; let $\tilde y$ be its end. This defines a map $y\longrightarrow \tilde y$, which is by construction continuous, and **defines a homeomorphism between U and a connected component of $\varphi^{-1}(U)$ containing $\tilde x$.** ■

Intrinsic metric (reminder)

DEFINITION: A metric d on M us called **intrinsic metric**, or **path metric**, if $d(x,y) = \inf_\gamma L_d(\gamma)$, where the infimum is taken over all rectifiable paths γ connecting x to y.

DEFINITION: Let M be a topological space. **A length structure** on M is a class $\mathcal{C}$ of admissible paths together with **a length functional** $L: \mathcal{C} \longrightarrow \mathbb{R}^{\geqslant 0} \cup \infty$ which satisfies the following axioms.
1. **Additivity:** for any path $\gamma: [a,c] \longrightarrow M$ and any $b \in [a,c]$, one has $L(\gamma) = L(\gamma|_{[a,b]}) + L(\gamma|_{[b,c]})$.
2. **The length is continuous as a function of the ends:** for any path $\gamma: [a,c] \longrightarrow M$, **the function $L(\gamma\Big|_{[a,b]})$ depends on $b \in [a,c]$ continuously.**
3 **Invariance under reparametrizations:** for any homeomorphism $\varphi: [a,b] \longrightarrow [a,b]$ and any admissible path $\gamma: [a,c] \longrightarrow M$ such that $\varphi \circ \gamma$ is also admissible, one has $L(\gamma) = L(\varphi \circ \gamma)$.
4. **Compatibility with the topology:** for any point $x \in M$, and any closed set $Z \subset M$, such that $x \notin Z$, there is a number $\varepsilon > 0$ such that $L(\gamma) > \varepsilon$ for any admissible path connecting Z to x.

DEFINITION: Let M be a topological space equipped with a length structure L. The **path metric** d_L associated with L is defined as $d_L(x,y) := \inf_\gamma L(\gamma)$, where the infimum is taken over all admissible paths connecting x tp y.
CLAIM: **d_L is an intrinsic metric**.

Cone structures

DEFINITION: Let X be a locally G-homogeneous space, and $C \subset \operatorname{Tot}(TX)$ a G-invariant, $\mathbb{R}$-invariant subset. We say that C is **a cone structure** if C generates a sub-bundle $B \subset TX$ which satisfies the Chow-Rashevskii condition, and, moreover, for each $x \in M$ the intersection $C_x := C \cap T_x M$ is open and dense in its fiberwise closure $\overline{C}_x$, and the union of all $\overline{C}_x$ the closure of C.

DEFINITION: An **admissible path** for a cone structure is a piecewise smooth path tangent to C everywhere. Fix a reference Riemannian metric g on X, and define **the cone metric** $d_C(x, y)$ as as infimum of the Riemannian length $L(\gamma)$, for all admissible paths connecting x to y which are tangent to C.

THEOREM: (D. Korshunov)
The cone metric is sub-Finsler.

Proof: `https://arxiv.org/abs/2410.18255` ■

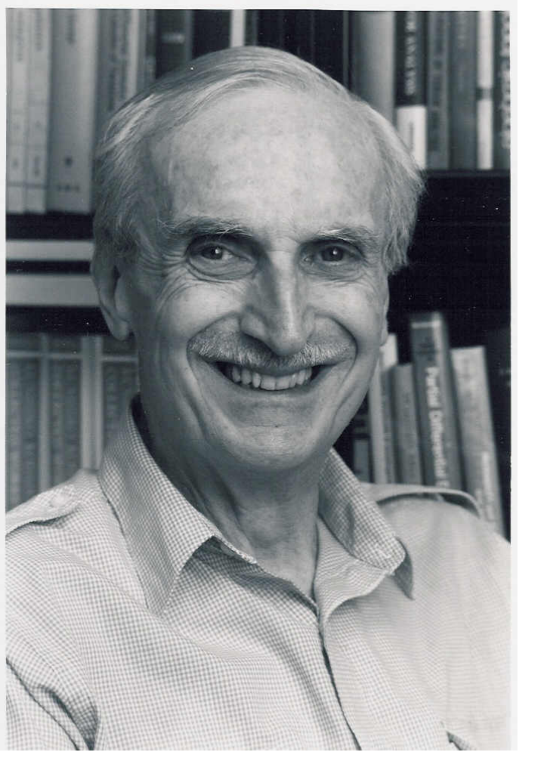
Cone structures and coverings

Proposition 1: Let $\varphi : \tilde{M} \longrightarrow M$ be a local diffeomorphism of manifolds equipped with compatible cone structures. Assume that φ has path lifting property for all admissible paths. **Then φ is a covering.**

Proof. Step 1: Since M is sub-Finsler, it is locally compact. Replacing M by a closure $\overline{B}_r(x)$ of a sufficiently small open ball $B_r(x)$ with center in $x \in M$, we may assume that M is complete and compact. Let r be the supremum of all $r' > 0$ such that $\overline{B}_{r'}(x)$ admits a lifting $\tilde{\overline{B}}_{r'}(x)$ to $\tilde{M}$ passing through $\tilde{x}$. Then the open ball $B_r(x)$ admits a lifting to $\tilde{M}$ containing $\tilde{x}$. Any point y in the boundary $\partial\overline{B}_r(x)$ is connected to x by a geodesic $u: [0,r] \longrightarrow M$ such that $u([0,r[) \subset B_r(x)$. Therefore, the lift of u to $\tilde{\overline{B}}_{r'}(x)$ contains y. **Applying this to all $y \in \partial\overline{B}_r(x)$, we lift $\overline{B}_r(x)$ to $\tilde{M}$.**

Step 2: We are going to show that $r = \infty$; to prove this **we assume, on contrary, that the boundary of $\overline{B}_r(x)$ is non-empty, and prove that $\overline{B}_r(x)$ can be lifted with its ε-neighbourhood.** Since $\overline{B}_r(x)$ compact, its boundary $\partial\overline{B}_r(x)$ is compact. For each $y \in \partial\overline{B}_r(x)$, let $\lambda(y)$ be the supremum of all λ' such that the lift of $\overline{B}_r(x)$ is extended to the lift of $\overline{B}_r(x) \cap B_{\lambda'}(y)$; since φ is a local diffeomorphism, $\lambda(y) > 0$. This function is by construction Lipschitz, hence continuous, hence its infimum is $\varepsilon > 0$. Then the ball $B_{r+\varepsilon}(x)$ admits a lifting to $\tilde{M}$ containing $\tilde{x}$, and we came to a contradiction. ■

Hyperkähler manifolds: reminder



Eugenio Calabi, 1923-2023

DEFINITION: (E. Calabi, 1978)
Let (M,g) be a Riemannian manifold equipped with three complex structure operators $I,J,K: TM \longrightarrow TM$, satisfying the quaternionic relations

$$I^2 = J^2 = K^2 = IJK = -\operatorname{Id}.$$

Suppose that g is Kähler with respect to I, J, K. Then (M,I,J,K,g) is called **hyperkähler**.

REMARK: This is the same as $\mathcal{H}\text{ol}(M) \subset Sp(n)$.
Indeed, if $\mathcal{H}\text{ol}(M) \subset Sp(n)$, we have 3 complex structures $I,J,K: TM \longrightarrow TM$, such that $\nabla(I) = \nabla(J) = \nabla(K) = 0$, which implies that I,J,K are Kähler. Conversely, if I,J,K are Kähler, we have $\nabla(I) = \nabla(J) = \nabla(K) = 0$.

EXERCISE: Prove that **the form $\omega_J + \sqrt{-1}\,\omega_K$ is holomorphically symplectic on (M,I).**

Calabi-Yau theorem: reminder

DEFINITION: Let (M, I, ω) be a Kähler manifold. Its **Kähler class** is the cohomology class of ω in $H^{1,1}_{\mathbb{R}}(M, I)$.

DEFINITION: Let (M, I, ω) be a Kähler manifold with trivial canonical bundle, and $\Omega \in \Lambda^{n,0}_I(M)$ its non-degenerate holomorphic section. We say that (M, I) is **Ricci-flat** if $|\Omega| = const$, where $|\cdot|$ denotes the metric on $\Lambda^{n,0}_I(M)$ induced by the Kähler metric.

THEOREM: (Calabi-Yau)
Let (M, Ω) be compact holomorphically symplectic manifold of Kähler type, and $[\omega] \in H^{1,1}_{\mathbb{R}}(M, I)$ a Káhler class. **Then there exists a unique Ricci-flat metric g on (M, I) such that its Kähler class is equal to $[\omega]$.**

THEOREM: (Calabi)
A Kähler metric on a compact holomorphically symplectic manifold **is hyperkähler if and only if it is Ricci-flat.**

COROLLARY: A compact holomorphically symplectic manifold of Kähler type **admits a unique hyperkähler metric in each Kähler class.**

The hyperkähler Teichmüller space (reminder)

DEFINITION: Let Teich_h be the Teichmüller space of conformal classes of hyperkähler metrics on a K3 surface M, that is, the space of all hyperkähler metrics on M up to $\mathbb{R}^{>0} \times \operatorname{Diff}_0$-action. We will call Teich_h **"the hyperkähler Teichmüller space"**.

REMARK: It is not hard to see that the topology on Teich_h, identified with the set of hyperkähler metrics of diameter 1, is induced by d_{GH}, **and therefore Teich_h is Hausdorff.**

Oswald Teichmüller, 1913-1943

The hyperkähler period map (reminder)

REMARK: Let M be a compact Riemannian manifold, and $\mathcal{H}^2(M,\mathbb{R})$ be the space of harmonic 2-forms, tacitly identified with the second cohomology. Since the Hodge star operator commutes with the Laplacian, it preserves $\mathcal{H}^2(M,\mathbb{R})$. Denote by $H^+(M,\mathbb{R})$ the set of $*$-invariant harmonic forms, and $H^-(M,\mathbb{R})$ the set of $*$-anti-invariant harmonic forms; then $H^2(M) = H^+(M) \oplus H^-(M)$, and the intersection form is positive on $H^+(M)$ and negative on $H^-(M)$. This decomposition **defines a map from the space of all Riemannian metrics to the Grassmanian of all $\dim H^+(M)$-dimensional positive subspaces in $H^2(M,\mathbb{R})$.**

REMARK: To simplify the language, in the sequel **we will identify hyperkähler structures (I,J,K,g) and $(I,J,K,\lambda g)$, where λ is a constant**. For a K3, **the triple (I,J,K) determines g uniquely up to a constant**, hence we will say "hyperkähler structure I,J,K" signifying "(I,J,K,g) up to a constant multiplier".

DEFINITION: Given a hyperkähler structure I,J,K on a K3 surface, consider the decomposition $\Lambda^2(M) = \Lambda^+(M) \oplus \Lambda^-(M)$. The bundle $\Lambda^+(M)$ is trivialized by parallel sections $\omega_I, \omega_J, \omega_K$, which generate the subspace $H^+(M,\mathbb{R}) \subset \mathcal{H}^2(M,\mathbb{R})$. Let $\operatorname{Gr}_{+++}(H^2(M,\mathbb{R}))$, or simply Gr_{+++}, be the Grassmannian of 3-dimensional positive oriented subspaces in $H^2(M,\mathbb{R})$. Define **the hyperkähler period map** as the map $\operatorname{Per}_h : \operatorname{Teich}_h \longrightarrow \operatorname{Gr}_{+++}(H^2(M,\mathbb{R}))$ taking I,J,K to the corresponding space $H^+(M) = \langle \omega_I, \omega_J, \omega_K \rangle$.

Hyperkähler local Torelli (reminder)

DEFINITION: Let $\mathbb{P}\mathrm{er}_h \subset \mathrm{Gr}_{+++}$ be the set of all $W \in \mathrm{Gr}_{+++}(H^2(M,\mathbb{R}))$ such that $W^\perp$ does not contain (-2)-classes. We call $\mathbb{P}\mathrm{er}_h$ **the hyperkähler period space** of a K3 surface M.

PROPOSITION: Let M be a K3 surface. Consider the hyperkähler period map $\mathrm{Per}_h: \mathrm{Teich}_h \longrightarrow \mathrm{Gr}_{+++}$. **Then $\mathrm{Per}_h(g) \in \mathbb{P}\mathrm{er}_h$ for any hyperkähler structure g.**

Proof: Lecture 22.

PROPOSITION: The hyperkähler period map $\mathrm{Per}_h: \mathrm{Teich}_h \longrightarrow \mathbb{P}\mathrm{er}_h$ **is locally a homeomorphism.**

Proof: Lecture 22.

THEOREM: The hyperkähler period map $\mathrm{Per}_h: \mathrm{Teich}_h \longrightarrow \mathbb{P}\mathrm{er}_h$ **is a homeomorphism** for any connected component of Teich_h.

Proof: Later today.

Pencils of 3-spaces (reminder)

DEFINITION: Let (H,q) be a quadratic vector space of signature $(3,n)$, and $W_1, W_2 \in \operatorname{Gr}_{+++}$ 3-spaces such that $\dim W_1 \cap W_2 = 3$. The space $U = W_1 + W_2$ is 4-dimensional, because $W_1 \cap W_2$ is 2-dimensional. All 3-spaces in U are lines in U^*, that is, points in $\mathbb{P}U^*$; this equivalence can be obtained by associating with W the set $\operatorname{Ann}(W)$ of all $\lambda \in U^*$ vanishing on W. The points $W_1, W_2 \in \mathbb{P}U^*$ are connected by a unique line $\mathbb{P}l \subset \mathbb{P}U^*$, where $l \subset U^*$ is a 2-dimensional subspace obtained as by $l = \operatorname{Ann}(W_1) + \operatorname{Ann}(W_2)$. The corresponding family of 3-spaces is called **a 1-dimensional pencil of 3-spaces in U**.

DEFINITION: The **positive segment** connecting W_1 and W_2 is the only segment in $\mathbb{P}l = \mathbb{R}P^1$ connecting W_1 to W_2 and consisting of $u \in \mathbb{P}l$ with $q(u,u) > 0$ (that is, without sign changes).

The subtwistor cone structure

Let M be a K3, Gr_{+++} the corresponding Grassmann space, and $C \subset T\operatorname{Gr}_{+++}$ the set of all vectors tangent to subtwistor segments.

CLAIM: Let $W \in \operatorname{Gr}_{+++}$ be a point, and $C_W := T_W \operatorname{Gr}_{+++} \cap C$. Denote by $\overline{C}_W$ its closure. Then $\overline{C}_W \subset T_W \operatorname{Gr}_{+++} = \operatorname{Hom}(W, W^\perp)$ **is the set of all matrices of rank 1** in $\operatorname{Hom}(W, W^\perp)$. Moreover, **the union of all $\overline{C}_W$ is the closure of C.**

Proof: Lecture 23. ■

REMARK: This implies that C **defines a cone structure on** Gr_{+++}**.** We call it **the subtwistor cone structure.**

DEFINITION: A **subtwistor chain** is a path obtained by concatenating subtwistor segments. A **subtwistor path** is an admissible path for the subtwistor cone structure.

REMARK: From Korshunov's proof it is apparent that the length structure defined on the class of subtwistor chains defines the same metric as on subtwistor paths.

Path lifting for subtwistor cone structure

PROPOSITION: Consider the hyperkähler period map

$$\operatorname{Per}_h: \operatorname{Teich}_h \longrightarrow \operatorname{Gr}_{+++}(H^2(M,\mathbb{R})).$$

Then Per_h has path lifting property for subtwistor chains.

Proof: Each subtwistor segment is obtained by modifying the Kähler class ω_I inside the positive cone associated with the complex structure I, where I is determined by the 2-plane $\langle \operatorname{Re}\Omega, \operatorname{Im}\Omega\rangle$, with orthogonal complement not containing (-2)-classes. For such I, the Kähler cone is the positive cone, hence the subtwistor segment corresponds to a linear path connecting two points in the Kähler cone; such segment can clearly be lifted. ■

COROLLARY: The hyperkähler period map $\operatorname{Per}_h: \operatorname{Teich}_h \longrightarrow \mathbb{P}\mathrm{er}_h$ is a covering.

Proof: By Proposition 1, any local diffeomorphism which has path lifting property for admissible paths is a covering. ■

The hyperkähler global Torelli theorem

THEOREM: The hyperkähler period map $\operatorname{Per}_h : \operatorname{Teich}_h \longrightarrow \mathbb{P}\mathrm{er}_h$ for K3 surfaces **is a homeomorphism** for any connected component of Teich_h.

Proof. Step 1: Since $\operatorname{Per}_h : \operatorname{Teich}_h \longrightarrow \mathbb{P}\mathrm{er}_h$ is a covering, **theorem would follow if we prove that $\pi_1(\mathbb{P}\mathrm{er}_h) = 0$.**

Step 2: Let $\mathfrak{R} \subset H^2(M, \mathbb{Z})$ be the set of all (-2)-classes; for each $\eta \in \mathfrak{R}$, let $S_\eta \subset \operatorname{Gr}_{+++}$ be the set of all positive 3-spaces in $\eta^\perp$. By definition,

$$\mathbb{P}\mathrm{er}_h = \operatorname{Gr}_{+++} \Big\backslash \bigcup_{\eta \in \mathfrak{R}} S_\eta.$$

However, the dimension of the Grassmanian of 3-dimensional spaces in $\mathbb{R}^{n+3}$ is $\operatorname{Hom}(\mathbb{R}^3, \mathbb{R}^n) = 3n$, hence $\operatorname{codim} S_\eta = 3$. By transversality, removing a smooth codimension 3 submanifold does not change the fundamental group, hence $\pi_1(\mathbb{P}\mathrm{er}_h) = \pi_1(\operatorname{Gr}_{+++})$. **It remains to prove that $\pi_1(\operatorname{Gr}_{+++}) = 0$.**

Step 3: The symmetric space Gr_{+++} has non-positive sectional curvature, hence **it is contractible by Cartan-Hadamard theorem.** ■

EXERCISE: Prove that $\pi_1(\operatorname{Gr}_{+++}) = 0$ in as many different ways as you possibly can.