K3 surfaces

lecture 25: The global Torelli theorem for Teichmüller spaces

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Complex structures on a K3 surface

THEOREM: Let M be a complex manifold diffeomorphic to a K3 surface. **Then M is K3,** that is, satisfies $K_M = \mathcal{O}_M$.

Proof. Step 1: Since $b_1(M)$ is even, M is Kähler by Buchdahl-Lamari. Also, $c_2(M) = 24$ because it is its topological Euler characteristic. Since the signature of M is (3, 19), Hodge index theorem implies $\dim H^{2,0}(M) = 1$. Now, $b_1(M) = 0$ implies that $\chi(\mathcal{O}_M) = 2$ and $\operatorname{rk} H^0(K_M) = 1$. Riemann-Roch formula for surfaces gives $\chi(\mathcal{O}_M) = \frac{c_1^2+c_2}{12} = \frac{c_1^2}{2}+2$, hence $c_1^2 = 0$. This implies that $\chi(K_M^{\otimes i}) = 2$ for all i. **If we prove that $\operatorname{rk} H^0(K_M^{\otimes i}) > 0$ for all i, it would imply that K_M is trivial.**

Step 2: Assume, by contradiction, that K_M is non-trivial. Since K_M is effective, $\operatorname{rk} H^0(K_M^{\otimes i}) = 0$ for all $i < 0$. Serre's duality gives $\operatorname{rk} H^0(K_M^{\otimes i}) = H^2(K_M^{\otimes -i+1})$ and $\operatorname{rk} H^1(K_M^{\otimes i}) = \operatorname{rk} H^1(K_M^{\otimes -i+1})$. Then $\chi(K_M^{\otimes i}) = 2$ implies that $H^0(K_M^{\otimes i}) \geqslant 2$ for all $i > 1$. The corresponding sections of $K_M^{\otimes i}$ don't intersect, because c_1^2, hence the line system $K_M^{\otimes i}$ is globally generated and defines a holomorphic map π to $\mathbb{C}P^n$. The bundle $K_M^{\otimes i}$ restricted to any fiber of π has degree 0, hence the fibers of π are 1-dimensional. **This implies that M is a surface of Kodaira dimension 1.**

Step 3: From Kodaira-Enriques classification it follows that all surfaces of Kodaira dimension 1 are elliptic with base a curve S of genus $\geqslant 1$. Therefore, **$\pi_1(M)$ surjects to $\pi_1(S)$ and $H_1(M)$ is infinite.** This brings a contradiction.

■

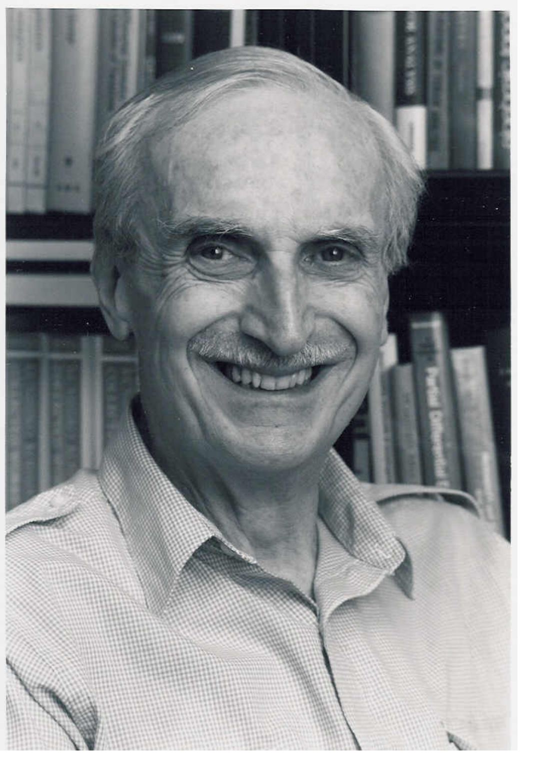
Hyperkähler manifolds: reminder

DEFINITION: (E. Calabi, 1978)
Let (M,g) be a Riemannian manifold equipped with three complex structure operators $I,J,K : TM \longrightarrow TM$, satisfying the quaternionic relations

$$I^2 = J^2 = K^2 = IJK = -\mathrm{Id}.$$

Suppose that g is Kähler with respect to I, J, K. Then (M,I,J,K,g) is called **hyperkähler**.

REMARK: This is the same as $\mathcal{H}\mathrm{ol}(M) \subset Sp(n)$.
Indeed, if $\mathcal{H}\mathrm{ol}(M) \subset Sp(n)$, we have 3 complex structures $I,J,K : TM \longrightarrow TM$, such that $\nabla(I) = \nabla(J) = \nabla(K) = 0$, which implies that I,J,K are Kähler. Conversely, if I,J,K are Kähler, we have $\nabla(I) = \nabla(J) = \nabla(K) = 0$.



Eugenio Calabi, 1923-2023

EXERCISE: Prove that **the form $\omega_J + \sqrt{-1}\,\omega_K$ is holomorphically symplectic on (M,I).**

Calabi-Yau theorem: reminder

DEFINITION: Let (M, I, ω) be a Kähler manifold. Its **Kähler class** is the cohomology class of ω in $H^{1,1}_{\mathbb{R}}(M, I)$.

DEFINITION: Let (M, I, ω) be a Kähler manifold with trivial canonical bundle, and $\Omega \in \Lambda^{n,0}_I(M)$ its non-degenerate holomorphic section. We say that (M, I) is **Ricci-flat** if $|\Omega| = const$, where $|\cdot|$ denotes the metric on $\Lambda^{n,0}_I(M)$ induced by the Kähler metric.

THEOREM: (Calabi-Yau)
Let (M, Ω) be compact holomorphically symplectic manifold of Kähler type, and $[\omega] \in H^{1,1}_{\mathbb{R}}(M, I)$ a Káhler class. **Then there exists a unique Ricci-flat metric g on (M, I) such that its Kähler class is equal to $[\omega]$.**

THEOREM: (Calabi)
A Kähler metric on a compact holomorphically symplectic manifold **is hyperkähler if and only if it is Ricci-flat.**

COROLLARY: A compact holomorphically symplectic manifold of Kähler type **admits a unique hyperkähler metric in each Kähler class.**

The hyperkähler Teichmüller space (reminder)

DEFINITION: Let Teich_h be the Teichmüller space of conformal classes of hyperkähler metrics on a K3 surface M, that is, the space of all hyperkähler metrics on M up to $\mathbb{R}^{>0} \times \operatorname{Diff}_0$-action. We will call Teich_h **"the hyperkähler Teichmüller space"**.

REMARK: It is not hard to see that the topology on Teich_h, identified with the set of hyperkähler metrics of diameter 1, is induced by d_{GH}, **and therefore Teich_h is Hausdorff.**

Oswald Teichmüller, 1913-1943

The hyperkähler period space (reminder)

REMARK: Let M be a compact Riemannian manifold, and $\mathcal{H}^2(M,\mathbb{R})$ be the space of harmonic 2-forms, tacitly identified with the second cohomology. Since the Hodge star operator commutes with the Laplacian, it preserves $\mathcal{H}^2(M,\mathbb{R})$. Denote by $H^+(M,\mathbb{R})$ the set of $*$-invariant harmonic forms, and $H^-(M,\mathbb{R})$ the set of $*$-anti-invariant harmonic forms; then $H^2(M) = H^+(M) \oplus H^-(M)$, and the intersection form is positive on $H^+(M)$ and negative on $H^-(M)$. This decomposition **defines a map from the space of all Riemannian metrics to the Grassmanian of all $\dim H^+(M)$-dimensional positive subspaces in $H^2(M,\mathbb{R})$.**

REMARK: To simplify the language, in the sequel **we will identify hyperkähler structures (I,J,K,g) and $(I,J,K,\lambda g)$, where λ is a constant**. For a K3, **the triple (I,J,K) determines g uniquely up to a constant**, hence we will say "hyperkähler structure I,J,K" signifying "(I,J,K,g) up to a constant multiplier".

The hyperkähler Torelli theorem (reminder)

DEFINITION: Given a hyperkähler structure I, J, K on a K3 surface, consider the decomposition $\Lambda^2(M) = \Lambda^+(M) \oplus \Lambda^-(M)$. The bundle $\Lambda^+(M)$ is trivialized by parallel sections $\omega_I, \omega_J, \omega_K$, which generate the subspace $H^+(M,\mathbb{R}) \subset \mathcal{H}^2(M,\mathbb{R})$. Let $\operatorname{Gr}_{+++}(H^2(M,\mathbb{R}))$, or simply Gr_{+++}, be the Grassmannian of 3-dimensional positive oriented subspaces in $H^2(M,\mathbb{R})$. Define **the hyperkähler period map** as the map $\operatorname{Per}_h: \operatorname{Teich}_h \longrightarrow \operatorname{Gr}_{+++}(H^2(M,\mathbb{R}))$ taking I, J, K to the corresponding space $H^+(M) = \langle \omega_I, \omega_J, \omega_K \rangle$.

THEOREM: The hyperkähler period map $\operatorname{Per}_h: \operatorname{Teich}_h \longrightarrow \mathbb{P}\mathrm{er}_h$ for K3 surfaces **is a homeomorphism** for any connected component of Teich_h.

Proof: Lecture 24. ■

Teichmüller space for symplectic structures (reminder)

DEFINITION: Let $\Gamma(\Lambda^2 M)$ be the space of all 2-forms on a manifold M, and $\mathrm{Symp} \subset \Gamma(\Lambda^2 M)$ the space of all symplectic 2-forms. We equip $\Gamma(\Lambda^2 M)$ with C^∞-topology of uniform convergence on compacts with all derivatives. Then $\Gamma(\Lambda^2 M)$ is a Frechet vector space, and Symp a Frechet manifold.

DEFINITION: **Teichmüller space of symplectic structures on M** is defined as a quotient $\mathrm{Teich}_s := \mathrm{Symp} / \mathrm{Diff}_0$, where Diff_0 is the isotopy group.

DEFINITION: Define **the period map** $\mathrm{Per}: \ \mathrm{Teich}_s \longrightarrow H^2(M,\mathbb{R})$ mapping a symplectic structure to its cohomology class.

THEOREM: (Moser, 1965)
The **Teichmüler space** Teich_s **is a manifold** (possibly, non-Hausdorff), and the **period map** $\mathrm{Per}: \ \mathrm{Teich}_s \longrightarrow H^2(M,\mathbb{R})$ **is locally a diffeomorphism.**

It is based on
THEOREM: ("Moser's lemma", 1965)
Let ω_t, $t \in S$ be a smooth family of symplectic structures, parametrized by a connected manifold S. Assume that the cohomology class $[\omega_t] \in H^2(M)$ is constant in t. **Then all ω_t are diffeomorphic.**

Torelli theorem for symplectic structures

DEFINITION: A symplectic structure on a K3 surface is **of Kähler type** if it is Kähler for some complex structure. **Conjecturally, all symplectic structures on a K3 surface are of Kähler type**. Let Teich_s be the Teichmüller space of symplectic structures of Kähler type, and $\mathbb{P}\mathrm{er}_s \subset H^2(M,\mathbb{R})$ the space of all vectors $v \in H^2(M,\mathbb{R})$ such that $\int_M v \wedge v > 0$.

THEOREM: The period map $\operatorname{Per}: \operatorname{Teich}_s \longrightarrow \mathbb{P}\mathrm{er}_s$ is a diffeomorphism on each connected component of Teich_s.

Proof: Let Teich_H be the space of all hyperkähler structures (g, I, J, K); clearly, it is fibered over Teich_h with the fiber $\mathbb{R}^{>0} \times SO(3)$. Let $P: \operatorname{Teich}_H \longrightarrow \operatorname{Teich}_s$ be the forgetful map putting (g, I, J, K) to ω_I. **Calabi-Yau theorem implies that P is surjective.** Indeed, any Kähler form can be deformed to a Ricci-flat Kähler form in the same cohomology class.

Step 2: From Torelli theorem for hyperkähler structures it follows that **the fiber $P^{-1}(\omega)$ of P is the space of pairs $x, y \in H^2(M)$ satisfying $x^2 = y^2 = \omega^2$, such that the space $\langle \omega, x, y\rangle^\perp$ contains no (-2)-classes.**

Step 3: Since the fibers of P are complements to subsets of codimension 2, they are connected. By Moser's lemma, for each $(M, I, J, K, g) \in P^{-1}(\omega)$ **the symplectic forms ω_I are diffeomorphic.**

Torelli theorem for symplectic structures (2)

THEOREM: The period map $\operatorname{Per}: \operatorname{Teich}_s \longrightarrow \mathbb{P}\mathrm{er}_s$ is a diffeomorphism on each connected component of Teich_s.

Step 4: Let $\mathbb{P}\mathrm{er}_H$ be the set of all orthogonal oriented triples $\omega_1, \omega_2, \omega_3$ with $\int_M \omega_i \wedge \omega_i = \int_M \omega_j \wedge \omega_j$ in $W \in \mathbb{P}\mathrm{er}_h \subset \operatorname{Gr}_{+++}(H^2(M,\mathbb{R}))$. It is $\mathbb{R}^{>0} \times SO(3)$-fibered over $\mathbb{P}\mathrm{er}_h$. Consider the diagram

$$\begin{array}{ccc} \operatorname{Teich}_H & \xrightarrow{P} & \operatorname{Teich}_s \\ \Big\downarrow \operatorname{Per}_H & & \operatorname{Per}_s \Big\downarrow \\ \mathbb{P}\mathrm{er}_H = \begin{array}{c} \{x,y,z \in H^2(M) \mid x^2 = y^2 = z^2 > 0, \\ \langle x,y,z\rangle^\perp \text{ contains no MBM classes}\} \end{array} & \xrightarrow{P'} & \{x \in H^2(M) \mid x^2 > 0\} \end{array}$$

The map Per_H is an isomorphism as follows from global Torelli for hyperkähler structures, and the fibers of P are identified with fibers of P' as follows from Moser's lemma and Step 3. Therefore, Per_s is injective. The rest of the arrows are surjective as shown, **hence Per_s is also surjective.** ■

Teichmüller space for complex structures (reminder)

DEFINITION: Let Teich be the Teichmüller space of complex structures of Kähler type on a K3 surface. The corresponding period space is denoted $\mathbb{P}\mathrm{er} := \{v \in \mathbb{P}H^2(M,\mathbb{C}) \quad | \quad \int_M v \wedge v = 0, \int_M v \wedge \overline{v} > 0\}$.

PROPOSITION: (local Torelli theorem)
Let Teich be the space of complex structures on a K3 surface, and Per : Teich $\longrightarrow \mathbb{P}\mathrm{er}$ the map taking (M,I) to the line $H^{2,0}(M) \subset H^2(M,\mathbb{C})$. **Then** Per **is a local diffeomorphism.**

Proof: Lecture 17. ■

Galina Tjurina (1938-1970)

REMARK: Torelli-type theorem for K3 surfaces was conjectured by André Weil, who included his ideas about K3 and Teichmüller theory in *"Final report on contract AF A8 (603-57)" (1958),* a report to a grant from american Air Force Office of Science Research, published only in 1979. Grauert attributes the local Torelli theorem for K3 surfaces to Andreotti and Weil (without reference; according to Grauert, the theorem was proven by Andreotti, but never published). Shafarevich attributes it to *G. N. Tjurina, On the deformation of complex structures of algebraic varieties Dokl. Akad. Nauk SSSR 152 (1963), 1316-1319*. Tjurina proved the local injectivity of the period map (actually, she proved it for all Calabi-Yau manifolds). For local surjectivity, she referred to *Kodaira, K.; Nirenberg, L.; Spencer, D. C. On the existence of deformations of complex analytic structures. Ann. of Math. (2) 68 (1958), 450-459.*

Galina Tjurina, 1938-1970

Kähler cone of K3 surfaces (reminder)

THEOREM: (Demailly-Păun)
Let M be a compact Kähler manifold, and $K \subset H^{1,1}(M, \mathbb{R})$ a subset consisting of all classes η such that $\int_Z \eta^p > 0$ for any p-dimensional complex subvariety $Z \subset M$. **Then the Kähler cone of M is one of the connected components of K.**

This implies

THEOREM: Let M be a K3 surface, and $\mathfrak{R}$ the set of all (-2)-classes. Denote by $\operatorname{Pos}(M)$ the component of $\{\omega \in H^{1,1}(M, \mathbb{R}) \ \mid \ \int_M \omega \wedge \omega > 0\}$ which contains Kähler classes (by Exercise 1, there is one and only one such component). **Then the Kähler cone $\operatorname{Kah}(M)$ is a connected component of the set $\operatorname{Pos}(M) \backslash \bigcup_{r \in \mathfrak{R}} r^{\perp}$.**

Proof: Lecture 21. ∎

Kähler chambers

DEFINITION: Let $V \in \mathbb{P}\mathrm{er} = \mathrm{Gr}_{++}(H^2(M,\mathbb{R}))$, and $I \in \mathrm{Teich}$ be a complex structure such that $\mathrm{Per}(I) = V$. The set $\mathrm{Kah}(I) \subset \mathrm{Pos}(I)$ is called **a Kähler chamber** of V. **The set of Kähler chambers** for V is the set of all $\mathrm{Kah}(I)$ for all $I \in \mathrm{Per}^{-1}(V)$.

THEOREM: Let $V \in \mathbb{P}\mathrm{er} = \mathrm{Gr}_{++}(H^2(M,\mathbb{R}))$, and $\mathfrak{R}_V$ be the set of all (-2)-classes orthogonal to V. Let $S_V := \bigcup_{\eta\in\mathfrak{R}_V}$. Then the Kähler chambers of V are connected components of $\mathrm{Pos}(V)\backslash S_V$.

Proof. Step 1: Let $I \in \mathrm{Per}^{-1}(V)$ be a complex structure. **Its Kähler cone is the set of all classes $\omega \in \mathrm{Pos}(V)$ which are positive on effective (-2)-classes;** clearly, this set is one of the connected components of $\mathrm{Pos}(V)\backslash S_V$.

Step 2: Consider the forgetful map $\mathrm{Teich}_H \longrightarrow H^2(M,\mathbb{R}) \times \mathrm{Teich}$ taking (I,J,K,g) to (I,ω_I). Since $\mathrm{Teich}_H = \mathbb{P}\mathrm{er}_H$, this map is surjective on the set of all $v \in \mathrm{Pos}(V)\backslash S_V$. Therefore, **any of the connected components of $\mathrm{Pos}(V)\backslash S_V$ is realized as a Kähler cone for some $I \in \mathrm{Per}^{-1}(V)$.** ■

Global Torelli theorem for Teichmüller spaces

THEOREM: Let Teich be a connected component of the Teichmüller space of all complex structures on a K3 surface, and $\operatorname{Per}: \operatorname{Teich} \longrightarrow \mathbb{P}\mathrm{er}$ the period map. **Then Per is surjective, and bijective for all $V \in \mathbb{P}\mathrm{er} \subset \mathrm{Gr}_{++}(H^2(M,\mathbb{R}))$ such that $V^\perp$ does not contain (-2)-curves.**

Proof. Step 1: Consider the space $\mathbb{P}\mathrm{er}_1$ of all pairs $\{(W \times V) \in \mathbb{P}\mathrm{er}_h \times \mathbb{P}\mathrm{er} \mid V \subset W\}$. This space is S^2-fibered over $\mathbb{P}\mathrm{er}_h$, with the fiber being the set of all oriented 2-planes in $W \in \mathbb{P}\mathrm{er}_h$. Similarly, let Teich_1 be the set of all pairs $(\mathcal{H}, L) \in \mathrm{Teich}_h \times \mathrm{Teich}$, consisting of all hyperkähler structures $\langle I, J, K\rangle$ inducing L; this space is also S^2-fibered over Teich. This gives a commutative diagram:

$$\begin{array}{ccc} \mathrm{Teich}_1 & \xrightarrow{P} & \mathrm{Teich} \\ \Big\downarrow{\scriptstyle \mathrm{Per}_1} & & \Big\downarrow{\scriptstyle \mathrm{Per}} \\ \mathbb{P}\mathrm{er}_1 & \xrightarrow{\Psi} & \mathbb{P}\mathrm{er} \end{array}$$

Step 2: Torelli theorem for $\mathbb{P}\mathrm{er}_h$ implies that Per_1 is a diffeomorphism. Calabi-Yau theorem implies that the fiber $P^{-1}(I)$ of $\mathrm{Teich}_1 \xrightarrow{P} \mathrm{Teich}$ is projectivization of the Kähler cone $\mathrm{Kah}(I)$ of (M, I); the fiber of $\mathbb{P}\mathrm{er}_1 \xrightarrow{\Psi} \mathbb{P}\mathrm{er}$ is the positive cone $\mathrm{Pos}(I)$. Since Per_1 is a diffeomorphism, Ψ is one to one on points for which $\mathrm{Kah}(I) = \mathrm{Pos}(I)$; for other points, $\Psi^{-1}(V) = \mathfrak{K}_V$, where $\mathfrak{K}_V$ is the set of Kähler chambers. ■