K3 surfaces

lecture 26: Reflections in cohomology and Kähler chambers

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Complex structures on a K3 surface (reminder)

THEOREM: Let M be a complex manifold diffeomorphic to a K3 surface. **Then M is K3,** that is, satisfies $K_M = \mathcal{O}_M$.

Proof. Step 1: Since $b_1(M)$ is even, M is Kähler by Buchdahl-Lamari. Also, $c_2(M) = 24$ because it is its topological Euler characteristic. Since the signature of M is (3, 19), Hodge index theorem implies $\dim H^{2,0}(M) = 1$. Now, $b_1(M) = 0$ implies that $\chi(\mathcal{O}_M) = 2$ and $\operatorname{rk} H^0(K_M) = 1$. Riemann-Roch formula for surfaces gives $\chi(\mathcal{O}_M) = \frac{c_1^2 + c_2}{12} = \frac{c_1^2}{2} + 2$, hence $c_1^2 = 0$. This implies that $\chi(K_M^{\otimes i}) = 2$ for all i. **If we prove that $\operatorname{rk} H^0(K_M^{\otimes i}) > 0$ for all i, it would imply that K_M is trivial.**

Step 2: Assume, by contradiction, that K_M is non-trivial. Since K_M is effective, $\operatorname{rk} H^0(K_M^{\otimes i}) = 0$ for all $i < 0$. Serre's duality gives $\operatorname{rk} H^0(K_M^{\otimes i}) = H^2(K_M^{\otimes -i+1})$ and $\operatorname{rk} H^1(K_M^{\otimes i}) = \operatorname{rk} H^1(K_M^{\otimes -i+1})$. Then $\chi(K_M^{\otimes i}) = 2$ implies that $H^0(K_M^{\otimes i}) \geqslant 2$ for all $i > 1$. The corresponding sections of $K_M^{\otimes i}$ don't intersect, because c_1^2, hence the line system $K_M^{\otimes i}$ is globally generated and defines a holomorphic map π to $\mathbb{C}P^n$. The bundle $K_M^{\otimes i}$ restricted to any fiber of π has degree 0, hence the fibers of π are 1-dimensional. **This implies that M is a surface of Kodaira dimension 1.**

Step 3: From Kodaira-Enriques classification it follows that all surfaces of Kodaira dimension 1 are elliptic with base a curve S of genus $\geqslant 1$. Therefore, **$\pi_1(M)$ surjects to $\pi_1(S)$ and $H_1(M)$ is infinite.** This brings a contradiction.

■

Teichmüller space for complex structures (reminder)

DEFINITION: Let Teich be the Teichmüller space of complex structures of Kähler type on a K3 surface. The corresponding period space is denoted $\mathbb{P}\mathrm{er} := \{v \in \mathbb{P}H^2(M, \mathbb{C}) \quad | \quad \int_M v \wedge v = 0, \int_M v \wedge \overline{v} > 0\}$.

PROPOSITION: (local Torelli theorem)
Let Teich be the space of complex structures on a K3 surface, and Per : Teich $\longrightarrow \mathbb{P}\mathrm{er}$ the map taking (M, I) to the line $H^{2,0}(M) \subset H^2(M, \mathbb{C})$. **Then** Per **is a local diffeomorphism.**

Proof: Lecture 17. ■

Kähler chambers (reminder)

DEFINITION: Let $V \in \mathbb{P}\mathrm{er} = \mathrm{Gr}_{++}(H^2(M,\mathbb{R}))$, and $I \in \mathrm{Teich}$ be a complex structure such that $\mathrm{Per}(I) = V$. The set $\mathrm{Kah}(I) \subset \mathrm{Pos}(I)$ is called **a Kähler chamber** of V. **The set of Kähler chambers** for V is the set of all $\mathrm{Kah}(I)$ for all $I \in \mathrm{Per}^{-1}(V)$.

THEOREM: Let $V \in \mathbb{P}\mathrm{er} = \mathrm{Gr}_{++}(H^2(M,\mathbb{R}))$, and $\mathfrak{R}_V$ be the set of all (-2)-classes orthogonal to V. Let $S_V := \bigcup_{\eta\in\mathfrak{R}_V}$. Then the Kähler chambers of V are connected components of $\mathrm{Pos}(V)\backslash S_V$.

Proof. Step 1: Let $I \in \mathrm{Per}^{-1}(V)$ be a complex structure. **Its Kähler cone is the set of all classes $\omega \in \mathrm{Pos}(V)$ which are positive on effective (-2)-classes;** clearly, this set is one of the connected components of $\mathrm{Pos}(V)\backslash S_V$.

Step 2: Consider the forgetful map $\mathrm{Teich}_H \longrightarrow H^2(M,\mathbb{R}) \times \mathrm{Teich}$ taking (I,J,K,g) to (I,ω_I). Since $\mathrm{Teich}_H = \mathbb{P}\mathrm{er}_H$, this map is surjective on the set of all $v \in \mathrm{Pos}(V)\backslash S_V$. Therefore, **any of the connected components of $\mathrm{Pos}(V)\backslash S_V$ is realized as a Kähler cone for some $I \in \mathrm{Per}^{-1}(V)$.** ∎

Global Torelli theorem for Teichmüller spaces (reminder)

THEOREM: Let Teich be a connected component of the Teichmüller space of all complex structures on a K3 surface, and $\operatorname{Per} : \quad \operatorname{Teich} \longrightarrow \mathbb{P}\mathrm{er}$ the period map. **Then** Per **is surjective, and bijective for all** $V \in \mathbb{P}\mathrm{er} \subset \mathrm{Gr}_{++}(H^2(M,\mathbb{R}))$ **such that** $V^\perp$ **does not contain (-2)-curves.**

Proof. Step 1: Consider the space $\mathbb{P}\mathrm{er}_1$ of all pairs $\{(W \times V) \in \mathbb{P}\mathrm{er}_h \times \mathbb{P}\mathrm{er} \ \mid V \subset W\}$. This space is S^2-fibered over $\mathbb{P}\mathrm{er}_h$, with the fiber being the set of all oriented 2-planes in $W \in \mathbb{P}\mathrm{er}_h$. Similarly, let Teich_1 be the set of all pairs $(\mathcal{H}, L) \in \mathrm{Teich}_h \times \mathrm{Teich}$, consisting of all hyperkähler structures $\langle I, J, K\rangle$ inducing L; this space is also S^2-fibered over Teich. This gives a commutative diagram:

$$\begin{array}{ccc} \mathrm{Teich}_1 & \xrightarrow{P} & \mathrm{Teich} \\ \Big\downarrow \mathrm{Per}_1 & & \Big\downarrow \mathrm{Per} \\ \mathbb{P}\mathrm{er}_1 & \xrightarrow{\Psi} & \mathbb{P}\mathrm{er} \end{array}$$

Step 2: Torelli theorem for $\mathbb{P}\mathrm{er}_h$ implies that Per_1 is a diffeomorphism. Calabi-Yau theorem implies that the fiber $P^{-1}(I)$ of $\mathrm{Teich}_1 \xrightarrow{P} \mathrm{Teich}$ is projectivization of the Kähler cone $\mathrm{Kah}(I)$ of (M, I); the fiber of $\mathbb{P}\mathrm{er}_1 \xrightarrow{\Psi} \mathbb{P}\mathrm{er}$ is the positive cone $\mathrm{Pos}(I)$. Since Per_1 is a diffeomorphism, Ψ is one to one on points for which $\mathrm{Kah}(I) = \mathrm{Pos}(I)$; for other points, $\Psi^{-1}(V) = \mathfrak{K}_V$, where $\mathfrak{K}_V$ is the set of Kähler chambers. ■

Automorphisms of K3 surfaces acting trivially on $H^2(M)$

Proposition 1: Let ν be an automorphism of a K3 surface (M, I) acting trivially on $H^2(M, \mathbb{Z})$. **Then $\nu = \mathrm{Id}$.**

Proof. Step 1: By Calabi-Yau theorem, ν acts by hyperkähler isometries for any hyperkähler metric on (M, I). Using the same argument as in the proof of Torelli theorem for hyperkähler structures, **we can extend ν to a hyperkähler isometry of any other hyperkähler structure in Teich_h.**

Step 2: Then ν acts by an automorphism of a Kummer surface $M = \widetilde{X/\pm 1}$, where X is any complex 2-torus. When X is very non-algebraic (e. g. contains no complex curves), it is easy to see that $\mathrm{Aut}\, M$ **is identified with the group of $\{\pm 1\}$-invariant automorphisms of X.** Since ν acts trivially on $H^2(X)$, it is induced by a translation of the torus. Finally, translations which are $\{\pm 1\}$-invariant are translations by 2-torsion. The group G of such translations is clearly isomorphic to the group $(\mathbb{Z}/2)^4$. This group acts freely and transitively on 16 singular points of $X/\pm 1$. Therefore, all non-trivial elements of G act on $H^2(M)$ non-trivially. ■

The Torelli group and the monodromy group

DEFINITION: Let Γ be the mapping class group of a K3 surface M. **The Torelli group** Γ_0 is the kernel of the natural map $\Gamma \longrightarrow O(H^2(M,\mathbb{Z}))$. Fix a connected component Teich_I of the Teichmüller space. **The mondromy group** Γ_I is the subgroup of the mapping class group generated by monodromies of all holomorphic local systems associated with the complex structures in Teich_I.

REMARK: Clearly, **the group Γ_I is generated by all $A \in \Gamma$ such that $I_1 = A(I_1)$ for some $I_1 \in \operatorname{Teich}_I$.**

REMARK: The mapping class group acts on set of connected components of Teich transitively. Indeed, the quartics are dense in $\operatorname{Teich}/\Gamma$, hence they are dense in any of the components of Teich; however, the set of all quartics is connected.

Marked K3 surfaces

DEFINITION: A marking on a K3 surface M is an isomorphism $H^2(M,\mathbb{Z}) \cong \Lambda$, where $\Lambda = (-E_8)^2 \oplus (U_2)^3$ is the lattice of the second cohomology of M.

REMARK: Fix an isomorphism $H^2(M,\mathbb{Z}) \cong \Lambda$ on a smooth K3 surface, and let Teich be the corresponding Teichmüller space. Then each $I \in$ Teich corresponds to a complex structure on a marked K3 surface. Conversely, global Torelli theorem implies that **each marked K3 surface (M,I) is obtained from a point in a Teichmüller space.**

PROPOSITION: The space of marked K3 surfaces is naturally identified with the Teichmüller space of K3 surfaces.

Proof: Since the quartics are dense in any connected component of Teichmüller space of K3 surfaces, the Torelli group acts transitively on the connected components of Teich. This action is also free, because any automorphism of a K3 acting trivially on $H^2(M)$ is trivial (Proposition 1). Therefore, a K3 surface with a marking uniquely determines a point in Teich. ■

Generalized Dehn twist

DEFINITION: A **support** of a diffeomorphism $A: M \longrightarrow M$ is the set of al $m \in M$ such that $A \neq \mathrm{Id}$ in any neighbourhood of m. A diffeomorphism is **compactly supported** if its support is compact.

DEFINITION: We identify points of TS^n (total space of the tangent bundle to a sphere) with pairs $X, Y \in \mathbb{R}^n$, such that $|X| = 1$ and $Y \perp X$. Choose a smooth function $\theta: \mathbb{R}^{\geqslant 0} \longrightarrow [0, \pi]$ such that $\theta(t) = 0$ when t is small, and $\theta(t) = \pi$ when it is large. A **generalized Dehn twist** is a compactly supported diffeomorphism of TS^n which takes (X, Y) to $R_\theta(X, Y)(-X, -Y)$, where $R_\theta(X, Y)$ is the rotation by angle θ in the plane $\langle X, Y \rangle$.

REMARK: Another interpretation of generalized Dehn's twist implies that it is actually a symplectomorphism: using the standard metric, we identify TS^n and T^*S^n. Consider the geodesic flow on $TS^n = T^*S^n$; it is a Hamiltonian flow associated with the Hamiltonian $h(v) = |v|^2$, hence it is a symplectomorphism. It acts as -1 the sphere of radius π. **Glue together the composition of $(X, Y) \mapsto (X, -Y)$ and the geodesic flow on $\{(X, Y) \mid |Y| \leqslant \pi\}$ and the identity map on $\{(X, Y) \mid |Y| \geqslant \pi\}$.** This will give a symplectomorphism, which is homotopy equivalent to the generalized Dehn twist defined above **(do this as an exercise).**

Reflections in the mapping class group

Proposition 2: Let M be a K3 surface, and $v \in H^2(M,\mathbb{Z})$ a (-2)-class. Consider the corresponding reflection in cohomology, $r_v(x) := x + (r,x)r$, where $(r,x) = \int_M r \wedge x$ is the intersection form. Then there exists an element of the mapping class group **acting as r_v on $H^2(M)$.**

Proof. Step 1: Let M_1 be a deformation of M such that $\operatorname{Pic}(M_1) = \langle v \rangle$. The standard argument using the Riemann-Roch formula (Lecture 19) implies that either v or $-v$ is effective; since $\operatorname{Pic}(M_1)$ is generated by v, **the class $\pm v$ is represented by irredicible, and therefore smooth rational curve.**

Step 2: Consider the Dehn's twist associated with the 2-sphere representing v. It acts on $H^2(M)$ as r_v, as follows from **the Picard-Lefschetz formula** (`http://verbit.ru/IMPA/VHS-2024/slides-VHS-2024-09.pdf`). ■

The mapping class group of a K3 surface

EXERCISE: Let M be a K3 surface, and $\mathscr{P} \subset H^2(M,\mathbb{R})$ the space of all positive real vectors. **Prove that $\mathscr{P}$ is homotopy equivalent to S^2.** Denote by $O^+(H^2(M,\mathbb{R}))$ the index 2 subgroup of $O(H^2(M,\mathbb{R}))$ of all isometries preserving the generator of $H^2(\mathscr{P},\mathbb{Z})$. **Prove that $O^+(H^2(M,\mathbb{R}))$ is an index 2 subgroup of $O(H^2(M,\mathbb{R}))$.**

COROLLARY: (Borcea, Donaldson)
Let Mon be the image of the mapping class group of K3 in $O(H^2(M,\mathbb{Z}))$. **Then Mon is an index 2 subgroup obtained as the intersection $O(H^2(M,\mathbb{Z})) \cap O^+(H^2(M,\mathbb{R}))$.**

Proof: As shown by C. T. C. Wall, $O^+(H^2(M,\mathbb{Z}))$ is an index 2 subgroup in $O(H^2(M,\mathbb{Z}))$ generated by reflections r_v for all (-2)-classes v (*C. T. C. Wall, On the orthogonal groups of unimodular quadratic forms II, J. Reine Angew. Math. 213. (1963/64), 122-136.*). Each r_v belongs to Mon as follows from Proposition 2 (this was the original contribution of Borcea). Finally, $\mathrm{Mon} \subset O^+(H^2(M,\mathbb{Z}))$, as shown in *S. K. Donaldson, The orientation of Yang-Mills moduli spaces and 4-manifold topology, J. Diff. Geom. 26 (1987), 397-428.* ■

The Weyl group of a K3 surface

DEFINITION: Let (M,I) be a K3 surface, and $\mathfrak{R} \subset H^{1,1}(M,\mathbb{Z})$ the set of all (-2)-classes of type (1,1). **The Weyl group** of a K3 is the subgroup of $O^+(H^2(M,\mathbb{Z}))$ generated by the reflections r_v for all $v \in \mathfrak{R}$.

THEOREM: The Weyl group W of K3 surface **acts transitively on the set of Kähler chambers.**

Proof: Let Mon_I be the set of all $A \in O^+(H^2(M,\mathbb{Z}))$ preserving the Hodge structure on $H^2(M,\mathbb{R})$. From the explicit description of Kähler chambers it follows that Mon_I takes a Kähler chambers to a Kähler chambers. **Therefore, the Weyl group W takes a Kähler chamber to a Kähler chamber.**

Step 2: Let $K \subset \operatorname{Pos}(M,I)$ be a Kähler chamber, and v a (-2)-vector such that $v^\perp$ contains a face F of K. Then r_v takes K to a Kähler chamber adjacent to F. This way, one obtains every Kähler chamber adjacent to K from K. Iterating this construction, we obtain Kähler chambers adjacent to ones adjanent to K, and so on. **Therefore, $W \cdot K$ is the set of all Kähler chambers.** ■

COROLLARY: Let $I_1, I_2 \in \operatorname{Teich}$ be two points with the same periods, $\operatorname{Per}(I_1) = \operatorname{Per}(I_2)$. **Then (M,I_1) and (M,I_2) is the same K3 with different markings.**

Proof: The action of each generator of the Weyl group changes the marking on the same K3 surface. ■

Hodge-theoretic Torelli theorem

DEFINITION: Let M be a K3 surface. **A K3-type Hodge structure** on $H^2(M)$ is a decompositions $H^2(M,\mathbb{C}) = H^{2,0}(M) \oplus H^{1,1}(M) \oplus H^{0,2}(M)$ such that $\overline{H^{p,q}(M)} = H^{q,p}(M)$, the intersection pairing between $H^{q,p}(M)$ and $H^{p_1,q_1}(M)$ vanishes unless $p = q_1, q = p_1$, and $\int_M \Omega \wedge \overline{\Omega} > 0$ for any non-zero $\Omega \in H^{2,0}(M)$.

THEOREM: Let M be a K3 surface, Γ its mapping class group, and $\mathfrak{S}$ the set of all K3-type Hodge structures up to $O^+(H^2(M,\mathbb{Z}))$-action. **Then the natural map from $\operatorname{Teich}/\Gamma \longrightarrow \mathfrak{S}$ is bijective.** In other words, **the K3-type Hodge structures are in bijective correspondence with classes of isomorphism of K3 surfaces.**

Proof: Clearly, $\mathfrak{S} = \frac{\mathbb{P}\mathrm{er}}{O^+(H^2(M,\mathbb{Z}))}$, hence $\operatorname{Per}: \operatorname{Teich} \longrightarrow \mathbb{P}\mathrm{er}$ is reduced to $\operatorname{Per}: \operatorname{Teich}/\Gamma \longrightarrow \frac{\mathbb{P}\mathrm{er}}{O^+(H^2(M,\mathbb{Z}))}$. This map is surjective by global Torelli theorem. For any $V \in \mathbb{P}\mathrm{er}$, the set $\operatorname{Per}^{-1}(V)$ is identified with the set of Kähler chambers in $\operatorname{Pos}(V^\perp)$. The Weyl group acts transitively on the set of Kähler chambers. Therefore, **the set $\operatorname{Per}^{-1}(V)$ is mapped to the same point in $\operatorname{Teich}/\Gamma$.**

Step 2: This implies that all $I \in \operatorname{Teich}$ which are mapped to the same point in $\frac{\mathbb{P}\mathrm{er}}{O^+(H^2(M,\mathbb{Z}))}$ are related by the Γ-action. ∎

Automorphism group of a K3 surface

REMARK: Let (M,I) be a K3 surface. **Proposition 1 implies that the natural map** $\operatorname{Aut}(M,I) \longrightarrow O^+(H^2(M,\mathbb{Z}))$ **is injective.** In the sequel, **we will identify** $\operatorname{Aut}(M,I)$ **and its image in** $O^+(H^2(M,\mathbb{Z}))$**.**

THEOREM: Let (M,I) be a K3 surface. Then $\operatorname{Aut}(M,I) \subset O^+(H^2(M,\mathbb{Z}))$ **is the group of all** $\rho \in O^+(H^2(M,\mathbb{Z}))$ **which preserve the Hodge decomposition and the Kähler cone of** (M,I)**.**

Proof. Step 1: Clearly, any $\rho \in \operatorname{Aut}(M,I)$ preserves the Hodge decomposition and the Kähler cone. **It remains to prove the converse:** any $\rho \in O^+(H^2(M,\mathbb{Z}))$ preserving the the Hodge decomposition and the Kähler cone is induced by an automorphism of (M,I).

Step 2: Consider the action of ρ on the space $\operatorname{Teich}_H = \mathbb{P}\mathrm{er}_H$ of hyperkähler structures (I,J,K,g). This action is extended to the corresponding universal fibration, because points of Teich_H are hyperkähler structures on marked K3, and $O^+(H^2(M,\mathbb{Z}))$ just changes the marking.

Step 3: Let $\Psi: \operatorname{Teich}_H \longrightarrow \operatorname{Teich}$ take (I,J,K,g) to $I \in \operatorname{Teich}$. Since ρ preserves the Kähler cone and the Hodge decompositon, it takes $\Psi^{-1}(I)$ to $\Psi^{-1}(I)$. Therefore, ρ takes (I,J,K,g) to (I,J',K',g'); this map is by construction holomorphic. ■