Hodge theory

lecture 3: Frobenius theorem

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Integrability of almost complex structures (reminder)

DEFINITION: An almost complex structure I on a manifold is called **integrable** if any point of M has a neighbourhood U diffeomorphic to an open subset of $\mathbb{C}^n$, in such a way that the almost complex structure I is induced by the standard one on $U \subset \mathbb{C}^n$.

CLAIM: Let M be a manifold, and $I \in \operatorname{End}(TM)$ an integrable almost complex structure. Denote by $\mathcal{O}_M$ the sheaf of holomorphic functions on M. Then the ringed space $(M, \mathcal{O}_M)$ **is a complex manifold**.

Proof: Clear from the definition. ■

CLAIM: Complex structure on a manifold M uniquely determines an integrable almost complex structure, and is determined by it.

Proof: Complex structure on a manifold M is determined by the sheaf of holomorphic functions $\mathcal{O}_M$, and $\mathcal{O}_M$ is determined by I as explained above. Therefore, an integrable almost complex structure defines a complex structure. Conversely, every complex structure gives a sub-bundle in $\Lambda^{1,0}(M) = d\mathcal{O}_M \subset \Lambda^1(M, \mathbb{C})$, and **such a sub-bundle defines an almost complex structure by Remark 1.** ■

Distributions

DEFINITION: Distribution, or **a Pfaffian system** on a manifold is a sub-bundle $B \subset TM$.

REMARK: Let $\Pi : TM \longrightarrow TM/B$ be the projection, and $x, y \in B$ some vector fields. Then $[fx, y] = f[x, y] - D_y(f)x$. This implies that **$\Pi([x, y])$ is $C^\infty(M)$-linear as a function of x and y.**

DEFINITION: The map $[B, B] \longrightarrow TM/B$ we have constructed is called **Frobenius bracket** (or **Frobenius form**); it is a skew-symmetric $C^\infty(M)$-linear form on B with values in TM/B.

DEFINITION: A distribution is called **holonomic**, or **involutive**, or **integrable**, if its Frobenius form vanishes.

Formal integrability

DEFINITION: Let $I : TM \longrightarrow TM$ be an almost complex structure on M, and $TM \otimes_{\mathbb{R}} \mathbb{C} = T^{1,0}M \oplus T^{0,1}$ the Hodge decomposition. An almost complex structure I on (M, I) is called **formally integrable** if $[T^{1,0}M, T^{1,0}] \subset T^{1,0}$, that is, if $T^{1,0}M$ is involutive.

DEFINITION: The Frobenius form $\Psi \in \Lambda^{2,0}M \otimes TM$ is called **the Nijenhuis tensor**.

CLAIM: An integrable almost complex structure **is always formally integrable.**

Proof: Locally, the bundle $T^{1,0}(M)$ is generated by d/dz_i, where z_i are complex coordinates. These vector fields commute, hence satisfy $[d/dz_i, d/dz_j] \in T^{1,0}(M)$. This means that the Frobenius form vanishes. ■

THEOREM: (Newlander-Nirenberg) A complex structure I on M is integrable if and only if it is formally integrable.

Proof: (real analytic case) next lecture.

Smooth submersions

DEFINITION: Let $\pi: M \longrightarrow M'$ be a smooth map of manifolds. This map is called **submersion** if at each point of M the differential $D\pi$ is surjective, and **immersion** if it is injective.

CLAIM: Let $\pi: M \longrightarrow M'$ be a submersion. Then each $m \in M$ has a neighbourhood $U \cong V \times W$, where V, W are smooth and $\pi|_U$ **is a projection of $V \times W = U \subset M$ to $W \subset M'$ along V.**

Proof: Follows from the inverse function theorem. ■

THEOREM: ("Ehresmann's fibration theorem")
Let $\pi: M \longrightarrow M'$ be a smooth submersion of compact manifolds. **Prove that π is a locally trivial fibration.**

Proof: Next slide.

DEFINITION: Vertical tangent space $T_\pi M \subset TM$ of a submersion $\pi: M \longrightarrow M'$ is the kernel of $D\pi$.

Ehresmann connections

DEFINITION: Let $\pi: M \longrightarrow Z$ be a smooth submersion, with $T_\pi M$ **the bundle of vertical tangent vectors** (vectors tangent to the fibers of π). An **Ehresmann connection** on π is a sub-bundle $T_{\text{hor}}M \subset TM$ such that $TM = T_{\text{hor}}M \oplus T_\pi M$. The **parallel transport** along the path $\gamma: [0,a] \longrightarrow Z$ associated with the Ehresmann connection is a diffeomorphism

$$V_t: \pi^{-1}(\gamma(0)) \longrightarrow \pi^{-1}(\gamma(t))$$

smoothly depending on $t \in [0,a]$ and satisfying $\frac{dV_t}{dt} \in T_{\text{hor}}M$.

CLAIM: Let $\pi: M \longrightarrow Z$ be a smooth fibration with compact fibers. Then **the parallel transport, associated with the Ehresmann connection, always exists.**

Proof: Follows from existence and uniqueness of solutions of ODEs. ∎

Foliations

Frobenius Theorem: Let $B \subset TM$ be a sub-bundle. Then B is involutive if and only if each point $x \in M$ has a neighbourhood $U \ni x$ and **a smooth submersion $U \xrightarrow{\pi} V$ such that B is its vertical tangent space: $B = T_\pi M$.**

REMARK: The implication **"$B = T_\pi M$" $\Rightarrow$ "Frobenius form vanishes"** is clear because of local coordinate form of the submersions.

DEFINITION: The fibers of π are called **leaves**, or **integral submanifolds** of the distribution B. Globally on M, **a leaf of B** is a maximal connected manifold $Z \hookrightarrow M$ which is immersed to M and tangent to B at each point. A distribution for which Frobenius theorem holds is called **integrable**. If B is integrable, the set of its leaves is called **a foliation**. The leaves are manifolds which are immersed to M, but not necessarily closed.

REMARK: To prove the Frobenius theorem for $B \subset TM$, **it suffices to show that each point is contained in an integral submanifold.** In this case, the smooth submersion $U \xrightarrow{\pi} V$ is the projection to the leaf space of B.

The history of Frobenius theorem

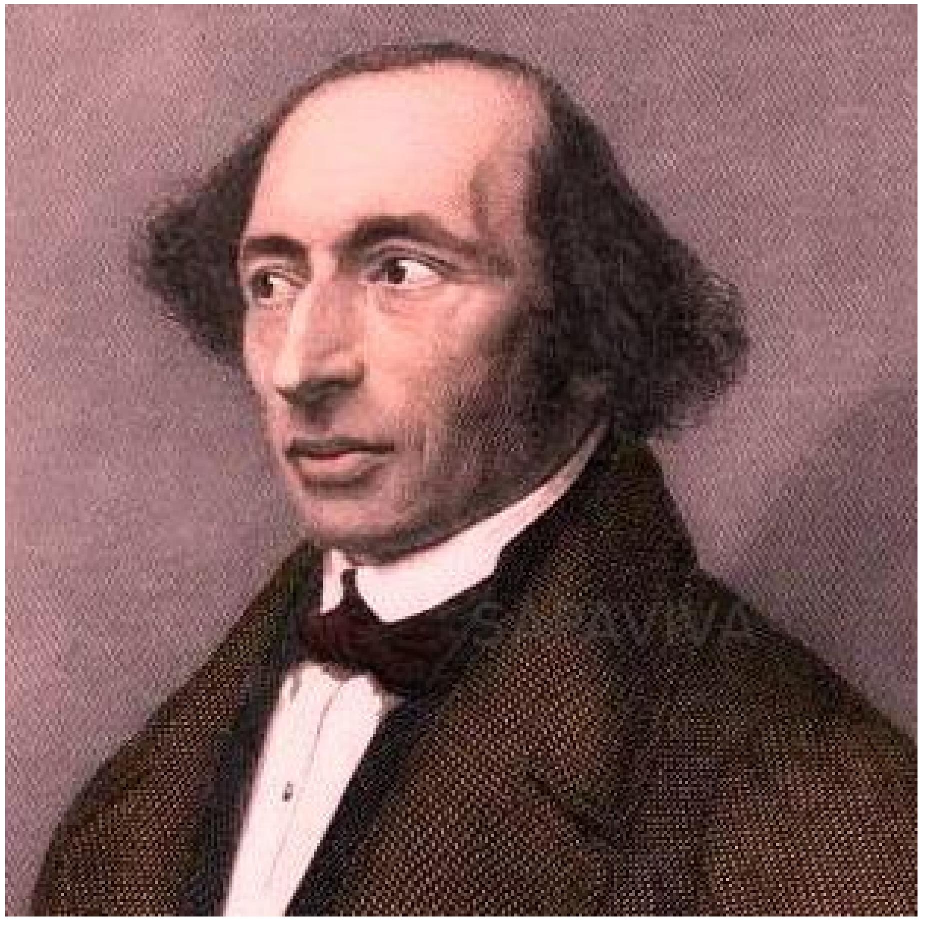
History of Frobenius theorem is explained in *"The Mathematics of Frobenius in Context: A Journey Through 18th to 20th Century Mathematics (Sources and Studies in the History of Mathematics and Physical Sciences)", by Thomas Hawkins.*

The name "Frobenius theorem" is due to Élie Cartan (1922). Before that it was known as "Pfaff's problem". It was a problem of having "sufficiently many" solutions for a system of differential equations.

In "generic" case was solved by Clebsch (1866), who generalized a weaker result of Jacobi (1837). In 1877, Frobenius gave an equivalent reformulation of Pfaff's problem and solve it, also taking care of the "non-generic" cases omitted by Clebsch.

The technical part of the argument of Frobenius is due to Heinrich Wilhelm Feodor Deahna (1815-1844), who published a version of solution of Pfaff problem in *Crelle's Journal* in 1844.

Carl Gustav Jacobi (1804-1851), Alfred Clebsch (1833-1872)



Carl Gustav Jacobi,
1804-1851

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Ferdinand Georg Frobenius (1849-1917)

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Frobenius theorem (1)

Proof of the Frobenius theorem. Step 1: Suppose that G is a Lie group acting on a manifold M. Assume that the vector fields from the Lie algebra of G generate a sub-bundle $B \subset TM$. **Then B is integrable, that is, Frobenius theorem holds of $B \subset TM$.** Indeed, the orbits of the G-action are tangent to $B \subset TM$.

Step 2: Let u, v be commuting vector fields on a manifold M, and e^{tu}, e^{tv} be corresponding diffeomorphism flows. **Then e^{tu}, e^{tv} commute.** This easily follows by taking a coordinate system such that u is the coordinate vector field **(do this as an exercise)**.

Step 3: The commutator of vector fields in B belongs to B, however, this does not immediately produce any finite-dimensional Lie algebra: it is not obvious that any subalgebra generated by such vector fields is finite-dimensional. To produce a Lie group with orbits tangent to B, **we need to find a collection $\xi_1, ..., \xi_k \in B$ of vector fields generating B and make sure that the $\xi_1, ..., \xi_k$ generate a finite-dimensional Lie algebra.**

Frobenius theorem (2)

Step 4: The statement of Frobenius Theorem is local, hence we may replace M be a small neighbourhood of a given point. **We are going to show that B locally has a basis of commuting vector fields.** By Step 2, these vector fields can be locally integrated to a commutative group action, and Frobenius Theorem follows from Step 1.

Step 5: Let $\sigma: M \longrightarrow M_1$ be a smooth submersion, $d\sigma: T_xM \longrightarrow T_{\sigma(x)}M_1$ its differential, and $v \in TM$ a vector field which satisfies

$$d\sigma(v)|_x = d\sigma(v)|_y \quad (*)$$

for any $x, y \in \sigma^{-1}(z)$ and any $z \in M_1$. In this case, the vector field $d\sigma(v)$ is well-defined on M_1. **Given two vector fields u and v which satisfy (*), we can easily check that the commutator $[u, v]$ also satisfies (*),** and, moreover, $d\sigma([u, v]) = [d\sigma(u), d\sigma(v)]$. Indeed, (*) is equivalent to existence of a vector field $\underline{x}$ on M_1 such that $\operatorname{Lie}_x(\sigma^* f) = \sigma^* \operatorname{Lie}_{\underline{x}} f$ for any $f \in C^\infty M_1$.

Frobenius theorem (3)

Step 6: Now we can finish the proof of Frobenius theorem. We need to produce, locally in M, a basis of commuting vector fields $\xi_i \in B$. **We start with producing (locally in M) an auxiliary submersion σ, with the fibers which are complementary to B.** To define such a submersion, we put coordinates locally on M, identifying M with an open subset in $\mathbb{R}^n$, and take a linear map $\sigma : M \longrightarrow M_1 = \mathbb{R}^{\dim B}$ such that $d\sigma : B|_x \longrightarrow T_{\sigma(x)}M_1$ is an isomorphism at some $x \in M$.

Step 7: Then $d\sigma : B|_x \xrightarrow{\sim} T_{\sigma(x)}M_1$ is an isomorphism in a neighbourhood of x; replacing M by a smaller open set, we may assume that $d\sigma : B|_x \xrightarrow{\sim} T_{\sigma(x)}M_1$ is an isomorphism everywhere on M. Let $\zeta_1, ..., \zeta_k$ be the coordinate vector fields on M_1.

Since $d\sigma : B|_x \longrightarrow T_{\sigma(x)}M_1$ is an isomorphism, there exist unique vector fields $\xi_1, ..., \xi_k \in B \subset TM$ such that $d\sigma(\xi_i) = \zeta_i$. By Step 5, $d\sigma([\xi_i, \xi_j]) = [\zeta_i, \zeta_j] = 0$. Since B is involutive, the commutator $[\xi_i, \xi_j]$ is a section of B. Now, the map $d\sigma : B|_x \longrightarrow T_{\sigma(x)}M_1$ is an isomorphism, and therefore the vanishing of $d\sigma([\xi_1, \xi_j])$ implies $[\xi_1, \xi_j] = 0$. **We have constructed a basis of commuting vector fields in B and finished the proof of Frobenius theorem.** ■