Hodge theory

lecture 12: Dolbeault cohomology of a 1-dimensional disk

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Stone-Weierstrass approximation theorem

DEFINITION: Let M be a topological space, and $\|f\| := \sup_M |f|$ **the sup-norm on functions**. C^0**-topology** on the space $C^0(M)$ of continuous, bounded real-valued functions is the topology defined by the sup-norm.

EXERCISE: Prove that $C^0(M)$ **with sup-norm is a complete metric space.**

DEFINITION: Let $A \subset C^0M$ be a subspace in the space of continuous functions. We say that A **separates the points** of M if for all distinct points $x, y \in M$, there exists $f \in A$ such that $f(x) \neq f(y)$.

THEOREM: (Stone-Weierstrass theorem)
Let $A \subset C^0M$ be a subring separating points, and $\overline{A}$ its closure. **Then** $\overline{A} = C^0M$**.**

Hilbert spaces

DEFINITION: Hilbert space over $\mathbb{C}$ is a complete, infinite-dimensional Hermitian space which is second countable (that is, has a countable dense set).

DEFINITION: Orthonormal basis in a Hilbert space H is a set of pairwise orthogonal vectors $\{x_\alpha\}$ which satisfy $|x_\alpha| = 1$, and such that H is the closure of the subspace generated by the set $\{x_\alpha\}$.

THEOREM: Any Hilbert space has a basis, and all such bases are countable.

Proof: A basis is found using Zorn lemma. If it's not countable, open balls with centers in x_α and radius $\varepsilon < 2^{-1/2}$ don't intersect, which means that the second countability axiom is not satisfied. ■

THEOREM: All Hilbert spaces are isometric.

Proof: Each Hilbert space has a countable orthonormal basis. ■

Fourier series

EXAMPLE: Let (M, μ) be a space with measure. Consider the space V of measurable functions $f: M \longrightarrow \mathbb{C}$ such that $\int_M |f|^2 \mu < \infty$. For each $f, g \in V$, the integral $\int f\bar{g}\mu$ is well defined, by Cauchy inequality: $\int |fg|\mu < \sqrt{\int_M |f|^2 \mu \int_M |g|^2 \mu}$. This gives a Hermitian form on V. Let $L^2(M)$ denote the completion of V with respect to this metric. It is called **the space of square-integrable functions on M**. Its elements are called **L^2-functions.**

CLAIM: ("Fourier series") Functions $e_k(t) = e^{2\pi\sqrt{-1}\,kt}$, $k \in \mathbb{Z}$ on $S^1 = \mathbb{R}/\mathbb{Z}$ **form an orthonormal basis in the Hilbert space $L^2(S^1)$**.

Proof. Step 1: Orthogonality is clear from $\int_{S^1} e^{2\pi\sqrt{-1}\,kt} dt = 0$ for all $k \neq 0$ (prove it).

Step 2: The space of Fourier polynomials $\sum_{i=-n}^{n} a_k e_k(t)$ is dense in the space of continuous functions on the circle by the Stone-Weierstrass approximation theorem. Therefore, the closure of the space of functions which admit Fourier series is $L^2(S^1)$. ■

Fourier series on a torus

REMARK: Let $t_1, ..., t_n$ be coordinates on $\mathbb{R}^n$. We can think of t_i as of angle coordinates on the torus $T^n = \mathbb{R}^n/\mathbb{Z}^n$, considered as a product of n copies of S^1. Consider the **Fourier monomials** $F_{l_1,...,l_n} := \exp(2\pi\sqrt{-1}\sum_{i=1}^n l_i t_i)$, where $l_1, ..., l_n$ are integers. Clearly,

$$L^2(T^n) \cong \underbrace{L^2(S^1)\hat{\otimes}L^2(S^1)\hat{\otimes}...\hat{\otimes}L^2(S^1)}_{n \text{ times}}.$$

where $\hat{\otimes}$ denotes the completed tensor product. This implies that the **Fourier monomials form a Hilbert basis in $L^2(T^n)$.**

REMARK: This also follows directly from the Stone-Weierstrass theorem.

THEOREM: Let V be a Hilbert space, $\operatorname{Map}(T^n, V)$ continuous maps, and $L^2(T^n, V)$ a completion of $\operatorname{Map}(T^n, V)$ with respect to the L^2-norm $|v|^2 = \int_{T^n} |v(x)|^2 dx$. Consider an orthonormal basis $u_1, ..., u_n, ...$ in V. Then **an orthonormal basis in $\operatorname{Map}(T^n, V)$ is given by monomial maps $F_{l_1,...,l_n} u_j$** taking $s \in T^n$ to $F_{l_1,...,l_n}(s)u_j$.

Proof: Orthonormality of the collection $\{F_{l_1,...,l_n} u_j\}$ is clear. To prove its completeness (that is, the density of the subspace generated by $\{F_{l_1,...,l_n} u_j\}$), notice that $\operatorname{Map}(T^n, V)$ is a completion of $\oplus_i \operatorname{Map}(T^n, V_i)$, where $V_i = \langle v_i \rangle$. Now, $\{F_{l_1,...,l_n} u_i\}$ is an orthonormal basis in $V_i = \operatorname{Map}(T^n, \mathbb{C})$. ■

Weight decomposition for $U(1)$-representations

EXERCISE: Let $\rho : U(1) \longrightarrow GL(V)$ be a finite-dimensional irreducible complex representation of the Lie group $U(1)$. **Prove that $\dim \mathbb{C} = 1$ and there exists $n \in \mathbb{Z}$ such that $t \in U(1) = \mathbb{R}/\mathbb{Z}$ acts on V as $\rho(t)(v) = e^{2\pi\sqrt{-1}\, nt}v$.**

DEFINITION: A representation of $U(1)$ with $\rho(t)(v) = e^{2\pi\sqrt{-1}\, nt}v$ is called **an irreducible weight n** representation.

DEFINITION: Let V be a Hermitian space (possibly infinitely-dimensional) equipped with an action of $U(1)$, and $V_k \subset V$ weight k representations, $k \in \mathbb{Z}$. The direct sum $\bigoplus V_k$ is called **the weight decomposition** for V if it is dense in V.

EXAMPLE: Let $L^2(S^1, W)$ the space of maps from S^1 to a Hermitian space W. We define $U(1)$-action on $L^2(S^1, W)$ by $\rho(t)(f) = R_t(f)$ where $R_t(f(x)) = f(x+t)$ shifts S^1 by t. Clearly, this is a Hermitian representation, and **its weight decomposition is its Fourier decomposition.**

Weight decomposition for $U(1)$-representations (2)

CLAIM: Let $\bigoplus V_k \subset V$ be the weight decomposition of a Hermitian representation ρ of $U(1)$. Then **any vector $v \in V$ can be decomposed onto a converging serie $v = \sum_{i\in\mathbb{Z}} v_i$, with $v_i \in V_i$.** This decomposition is called **the weight decomposition** for v.

Proof. Step 1: Clearly, all V_i are pairwise orthogonal; indeed, for any $t \in U(1)$ and $x_p \in V_p$, $x_q \in V_q$, $i \neq j$, we have

$$e^{2\pi\sqrt{-1}\,pt}(x_p, x_q) = (\rho(t)(x_p), x_q) = (x_p, \rho(-t)x_q) = \\ = (x_p, e^{-2\pi\sqrt{-1}\,qt}x_q) = e^{2\pi\sqrt{-1}\,qt}(x_p, x_q)$$

giving $p = q$ whenever $(x_p, x_q) \neq 0$.

Step 2: Let $\pi_i : V \longrightarrow V_i$ be the orthogonal projection. Then $|x|^2 \geqslant \sum_{i=-p}^{p} |\pi_i(x)|^2$ because orthogonal projection is always distance-decreasing. Therefore, the serie $\sum_{i\in\mathbb{Z}} \pi_i(x)$ converges. Its limit is a vector x' which satisfies $(x, u) = (x', u)$ for any $u \in \bigoplus_{k\in\mathbb{Z}} V_k$. Since $\bigoplus_{k\in\mathbb{Z}} V_k$ is dense in V, this implies $x = x'$. ■

Weight decomposition and Fourier series

LEMMA: Let W be a Hermitian representation of $U(1)$ admitting a weight decomposition. Then **any subquotient of W also admits a weight decomposition.**

Proof: This is clear for quotients. Any closed subspace $V \subset W$ gives a direct sum decomposition $W = V \oplus V^\perp$, hence it also can be realized as a quotient. ∎

LEMMA: Let $\rho : U(1) \longrightarrow U(W)$ be a Hermitian representation of $U(1)$, and $L^2(S^1, W)$ the space of maps from S^1 to W with the $U(1)$-action by translation as defined earlier. **Then W can be realized as a sub-representation of $L^2(S^1, W)$.**

Proof: For any $x \in W$ consider $\alpha_x \in L^2(S^1, W)$ taking $t \in U(1) = \mathbb{R}/\mathbb{Z}$ to $\rho(t)(x)$. Clearly, $x \mapsto \alpha_x$ **defines a homomorphism of representations.** ∎

THEOREM: Let W be a Hermitian representation of $U(1)$. **Then W admits a weight decomposition $W = \widehat{\bigoplus_{i\in\mathbb{Z}} W_i}$.**

Proof: We realize W as a subrepresentation in $L^2(S^1, W)$, and use the Fourier series to obtain the weight decomposition of $L^2(S^1, W)$. ∎

Weight decomposition for T^n-action

EXERCISE: Consider the n-dimensional torus T^n as a Lie group, $T^n = U(1)^n$. **Prove that any finite-dimensional Hermitian representation of T^n is a direct sum of 1-dimensional representations,** with action of T^n given by $\rho(t_1, ..., t_n)(x) = \exp(2\pi\sqrt{-1}\ \sum_{i=1}^n p_i t_i)x$, for some $p_1, ..., p_n \in \mathbb{Z}^n$, called **the weights** of the 1-dimensional representation.

DEFINITION: Let V be a Hermitian space (possibly infinitely-dimensional) equipped with an action of T^n, and $V_\alpha \subset V$ weight α representations, $\alpha \in \mathbb{Z}^n$. The direct sum $\bigoplus_{\alpha\in\mathbb{Z}^n} V_\alpha$ is called **the weight decomposition** for V if it is dense in V.

THEOREM: Let W be a Hermitian vector space. Then **the Fourier series provide the weight decomposition on $L^2(T^n, W)$.** ■

THEOREM: Let W be a Hermitian representation of T^n. **Then W admits a weight decomposition $V = \widehat{\bigoplus_{\alpha\in\mathbb{Z}^n} W_\alpha}$.**

Proof: We realize W as a subrepresentation in $L^2(T^n, W)$, and use the Fourier series to obtain the weight decomposition of $L^2(T^n, W)$. ■

Weight decomposition for T^n-action on differential forms

REMARK: Let M be a manifold with the T^n-action, and

$$\Lambda^*(M) = \widehat{\bigoplus}_{\alpha\in\mathbb{Z}^n}\Lambda^*(M)_{p_1,\dots,p_k}$$

be the weight decomposition on the differential forms. Then the **de Rham differential preserves each term $\Lambda^*(M)_{p_1,\dots,p_k}$.** Indeed, d **commutes with the action of the Lie algebra of T^n, and $\Lambda^*(M)_{p_1,\dots,p_k}$ are its eigenspaces.**

REMARK: Let $\alpha = \sum\alpha_{p_1,\dots,p_k}$ be the weight decomposition. **The forms $\alpha_{p_1,\dots,p_k}$ are obtained by averaging**

$$e^{2\pi\sqrt{-1}\,\sum_{i=1}^n p_i t_i}\alpha = \operatorname{Av}_{T^n} e^{2\pi\sqrt{-1}\,\sum_{i=1}^n -p_i t_i}\alpha$$

hence they are smooth.

De Rham cohomology and T^n-action

THEOREM: Let M be a smooth manifold, and T^n a torus acting on M by diffeomorphisms. Denote by $\Lambda^*(M)^{T^n}$ the complex of T^n-invariant differential forms. **Then the natural embedding $\Lambda^*(M)^{T^n} \hookrightarrow \Lambda^*(M)$ induces an isomorphism on de Rham cohomology.**

Proof. Step 1: Let $\alpha \in \Lambda^*(M)$ be a form and $\alpha = \sum \alpha_{p_1,...,p_n}$ its weight decomposition, with $\alpha_{p_1,...,p_n} \in \Lambda^*_{p_1,...,p_n}(M)$ a form of weight $p_1, ..., p_n$. Since T^n-action commutes with de Rham differential, these forms are closed when α is closed.

Step 2: Let $r_1, ..., r_n$ be the standard generators of the Lie algebra of T^n rescaled in such a way that $\operatorname{Lie}_{r_k}(\exp(2\pi\sqrt{-1}\ \sum_{i=1}^n p_i t_i)) = \sqrt{-1}\, p_k$, and $i_{r_k}: \Lambda^i(M) \longrightarrow \Lambda^{i-1}(M)$ the convolution operator. Since $\operatorname{Lie}_{r_k} = \{d, i_{r_k}\}$, we have $p_k \alpha_{p_1,...,p_n} = d(i_{r_k}\alpha_{p_1,...,p_n})$ whenever $\alpha_{p_1,...,p_n}$ is closed. Therefore, **all terms in the weight decomposition $\alpha = \sum \alpha_{p_1,...,p_n}$ are exact except $\alpha_{0,0,...,0}$.**

Step 3: In the direct sum decomposition of the de Rham complex

$$\Lambda^*(M) = \Lambda^*(M)^{T^n} \oplus \widehat{\bigoplus}_{p_1,...,p_k \neq (0,0,...,0)} \Lambda^*_{p_1,...,p_k}(M)$$

the second component has trivial cohomology, because Lie_{r_k} is invertible on $\bigoplus_{p_k \neq 0} \Lambda^*_{p_1,...,p_n}(M)$ **(deduce it from $p_k \alpha_{p_1,...,p_k} = d(i_{r_k}\alpha_{p_1,...,p_k})$)**, and Lie_{r_k}(closed form) is exact. ■

Constant forms on a torus

DEFINITION: Let $T^n = (S^1)^n$ be a compact torus equipped with a action on itself by shifts, and $\Lambda^*_{const}(M)$. the space of T^n-invariant forms on T^n. These forms are called **constant differential forms**. Clearly, **constant forms have constant coefficients in the usual (flat) coordinates on the torus.**

THEOREM: The natural embedding $\Lambda^*_{const}(T^n) \hookrightarrow \Lambda^*(T^n)$ **induces an isomorphism $\Lambda^*_{const}(T^n) = H^*(T^n)$.**

Proof: The embedding $\Lambda^*_{const}(T^n) = \Lambda^*(T^n)^{T_n} \hookrightarrow \Lambda^*(T^n)$ induces an isomorphism on cohomology, however, all constant forms are closed, hence $H^*(\Lambda^*_{const}(T^n), d) = \Lambda^*_{const}(T^n)$. ■

Holomorphic vector fields

DEFINITION: Let (M,I) be a complex manifold, and $X\in TM$ a real vector field. It is called **holomorphic** if $\operatorname{Lie}_X(I)=0$, that is, if the corresponding flow of diffeomorphisms is holomorphic.

CLAIM: Let (M,I) be a complex manifold, and $X\in TM$ a holomorphic vector field. **Then $X^c:=I(X)$ is also holomorphic, and commutes with X.**

LEMMA: Let X be a holomorphic vector field, and $X^c=I(X)$. **Then $\{d^c,i_X\}=-\operatorname{Lie}_{X^c}$.**

Proof: Using $\{IdI^{-1},i_X\}=I\{d,I^{-1}i_XI\}I^{-1}$, we obtain $\{d^c,i_X\}=-I\{d,i_{X^c}\}I^{-1}=I\operatorname{Lie}_{X^c}I^{-1}$. However, X^c is holomorphic, hence $I\operatorname{Lie}_{X^c}I^{-1}=\operatorname{Lie}_{X^c}$. ■

PROPOSITION: Let X be a holomorphic vector field, and $X^c=I(X)$. **Then $\{\bar\partial,i_X\}=\frac{1}{2}(\operatorname{Lie}_X-\sqrt{-1}\operatorname{Lie}_{X^c})$.**

Proof: $\bar\partial=\frac{1}{2}(d+\sqrt{-1}\,d^c)$, hence

$$\{\bar\partial,i_X\}=\frac{1}{2}\operatorname{Lie}_X+\sqrt{-1}\,\{d^c,i_X\}=\frac{1}{2}(\operatorname{Lie}_X-\sqrt{-1}\operatorname{Lie}_{X^c}).$$

■

Dolbeault cohomology of an elliptic curve

PROPOSITION: Let $X = \mathbb{C}/\mathbb{Z}^2$ be an elliptic curve, and $\Lambda^*(X) = \bigoplus_{\alpha\in\mathbb{Z}^2} \Lambda^*(X)_{p_1,p_2}$ its weight decomposition under the T^2-action. Consider the space T^2-invariant forms $\Lambda^*(X)^{T^2} = \Lambda^*(X)_{0,0}$. **Then the natural embedding $\Lambda^*(X)^{T^2} \hookrightarrow \Lambda^*(X)$ induces an isomorphism of Dolbeault cohomology.**

Proof: Let $\alpha \in \Lambda^*(X)_{p_1,p_2}$ be a $\overline{\partial}$-closed form, with $(p,q) \neq (0,0)$. Suppose, for example, that $p \neq 0$, and X is the generator of the corresponding component of the Lie algebra such that $\operatorname{Lie}_X \alpha = p\sqrt{-1}\,\alpha$. Since X^c belongs to the same Lie algebra, we have $\operatorname{Lie}_{X^c}(\alpha) = v\alpha$, where $v \in \sqrt{-1}\,\mathbb{R}$. Then

$$\frac{\sqrt{-1}\,p + v}{2}\alpha = \frac{1}{2}(\operatorname{Lie}_X - \sqrt{-1}\operatorname{Lie}_{X^c})\alpha = \{\overline{\partial}, i_X\}\alpha = \overline{\partial} i_X \alpha, \quad (***)$$

hence α is $\overline{\partial}$-exact. This implies that **$\overline{\partial}$ has no cohomology on**

$$\bigoplus_{p_1,p_2\neq(0,0)} \Lambda^*(X)_{p_1,p_2}.$$

■

Dolbeault cohomology of a disk

COROLLARY: Let $K \subset \mathbb{C}$ be a compact subset, K^0 its interior, and $\eta \in \Lambda^{0,1}(K^0)$ a form smoothly extending to a neighbourhood of K. **Then η is $\overline{\partial}$-exact.**

Proof: Choosing an appropriate lattice $\mathbb{Z}^2 \subset \mathbb{C}$, we may assume that K is a subset of an elliptic curve X. Since η extends to a neighbourhood of K, we can use partition of unity to extend it to a smooth form $\tilde{\eta}$ on X. Applying the weight decomposition $\tilde{\eta} = \sum_{\alpha \in \mathbb{Z}^2} \eta_\alpha$, we obtain that the form $\eta - \eta_{0,0}$ is $\overline{\partial}$-exact. However, the constant part $\eta_{0,0} = const \cdot dz \wedge d\overline{z} = const \cdot \overline{\partial}(\overline{z} dz)$ (for (1,1)-form) or $\eta_{0,0} = const \cdot d\overline{z} = const \cdot \overline{\partial}(\overline{z})$ for (0,1)-form is also $\overline{\partial}$-exact. ■

Poincaré-Dolbeault-Grothendieck lemma

DEFINITION: Polydisc D^n is a product of n discs $D \subset \mathbb{C}$.

THEOREM: (Poincaré-Dolbeault-Grothendieck lemma)
Let $\eta \in \Lambda^{p,q}(D^n)$, $q > 0$, be a $\overline{\partial}$-closed form on a polydisc, smoothly extended to a neighbourhood of its closure $\overline{D^n} \subset \mathbb{C}^n$. **Then η is $\overline{\partial}$-exact.**

We proved it for $n = 1$. Now we prove it for all n.

$\overline{\partial}$-homotopy operator on T^2

From now on, **1-dimensional complex torus is always $\mathbb{C}/\mathbb{Z}[\sqrt{-1}\,]$ and the n-dimensional complex torus T^{2n} is a product of n copies of $T^2 = \mathbb{C}/\mathbb{Z}[\sqrt{-1}\,]$.**

CLAIM: Let $\mu \in \Lambda^{p,q}(M)_{a,b}$ be a form of weight (a, b) on a torus $T^2 = \mathbb{C}/\mathbb{Z}[\sqrt{-1}\,]$, and X the coordinate vector field along the real axis. **Then $\{\overline{\partial}, i_X\}(\mu) = \frac{1}{2}(b + \sqrt{-1}\,a)$.**

Proof: $\{\overline{\partial}, i_X\} = \frac{1}{2}(\operatorname{Lie}_X - \sqrt{-1}\,\operatorname{Lie}_{X^c})$, and X^c is the coordinate vector field along the imaginary axis, acting on μ by multiplication by $\sqrt{-1}\,b$. ■

DEFINITION: Given $\mu = \sum_{a,b\in\mathbb{Z}^2} \mu_{a,b}$ define

$$P(\mu) := \sum_{(a,b)\neq(0,0)} 2(b + \sqrt{-1}\,a)^{-1}\mu_{a,b}.$$

The operator P **commutes with all operators which commute with the T^2-action on itself:** with d, d^c, i_X, i_{X^c}, etc.

COROLLARY: Then $\{\overline{\partial}, Pi_X\}) = \mu - \mu_{0,0}$. In particular, **if μ is $\overline{\partial}$-closed, we also have $\overline{\partial}P(i_X(\mu)) = \mu - \mu_0$.** ■

Homotopy operator γ_k on T^{2n}

Let $U \subset T^{2n}$ be a polydisk. Since U is contractible, all constant (p,q)-forms on a torus with $q > 0$ are $\bar\partial$-exact on U: $\bar\partial \bar z_i = \bar\partial(\bar z_i)$, which can be well defined on U because it is contractible.

For any disk $U \subset T^2$, fix a cutoff function ρ_ε which is 1 on U and 0 outside of a contractible ε-neighbourhood of $\overline{U}$. Consider the map $Q: \Lambda^{p,1}(T^2) \longrightarrow \Lambda^{p,0}(T^2)$ taking μ to $\mu_{0,0}$ and replacing any constant summand of form $\alpha \wedge \bar\partial \bar z_i$ by $\rho_\varepsilon \bar z_i \alpha$.

CLAIM: In these assumptions, **we have $\{\bar\partial, \gamma\}(\mu) = \mu$ on U for any form $\mu \in \Lambda^{p,1}(T^2)$,** where $\gamma(\alpha) = P(i_X(\alpha)) + Q(\mu)$.

Proof: If $\mu_{0,0} = 0$, we have $Q(\mu) = 0$, and this expression becomes $\{\bar\partial, P(i_X)\}) = \mu - \mu_{0,0}$ proven above. If $\mu = \mu_{0,0}$, it becomes $\bar\partial(Q(\mu))|_U = \mu$. ■

Corollary 1: Let $U \subset T^{2n}$ be a polydisk, and ρ_ε a cutoff function which is 1 on U and 0 outside of a contractible ε-neighbourhood of $\overline{U}$. We chose ρ_ε in such a way that $\operatorname{Lie}_{d/dx_i}(\rho_\varepsilon) = 0$ at any point $(x_1, ..., x_n)$ such that $|x_i| < 1$. Let γ_k denote the operator γ along the k-th component in $T^{2n} = (T^2)^n$, and $\bar\partial_k$ the $\bar\partial$ along this component. **Then $\{\bar\partial_k, \gamma_k\}(\mu) = \mu$ on U for any form μ divisible by $d\bar z_k$, and $\{\bar\partial_k, \gamma_l\}|_U = 0$ for $l \neq k$.** ■

Poincaré-Dolbeault-Grothendieck lemma

THEOREM: (Poincaré-Dolbeault-Grothendieck lemma)
Let $\eta \in \Lambda^{0,p}(D^n)$ be a $\overline{\partial}$-closed form on a polydisc, smoothly extended to a neighbourhood of its closure $\overline{D^n} \subset \mathbb{C}^n$. **Then η is $\overline{\partial}$-exact.**

We prove the following version of Poincaré-Dolbeault-Grothendieck.

THEOREM: Let $U \subset T^{2n}$ be a sufficiently small polydisk, and $\mu \in \Lambda^{p,q}(T^{2n})$ a form with $q > 0$ which is $\overline{\partial}$-closed on U. **Then there exists $\alpha \in \Lambda^{p,q-1}(T^{2n})$ such that $\overline{\partial}\alpha = \mu$ on U.**

Proof. Step 1: Let $\overline{\partial}_i : \Lambda^{p,q}(T^n) \longrightarrow \Lambda^{p,q+1}(T^n)$ be the operator $\alpha \longrightarrow d\overline{z}_i \wedge \frac{d}{d\overline{z}_i}\alpha$, where z_i is i-th coordinate on T^n. **Then $\overline{\partial} = \sum_i \overline{\partial}_i$.** Denote by γ_i the homotopy operator defined above. If $\alpha = d\overline{z}_i \wedge \beta$, one has $\{\overline{\partial}_i, \gamma_i\}(\alpha) = \alpha$. If α contains no monomials divisible by $d\overline{z}_i$, one has

$$\overline{\partial}_i\{\overline{\partial}_i, \gamma_i\}(\alpha) = \overline{\partial}_i\gamma_i\overline{\partial}_i(\alpha) = \{\overline{\partial}_i, \gamma_i\}\overline{\partial}_i\alpha = \overline{\partial}_i\alpha,$$

hence $\overline{\partial}_i(\alpha - \{\overline{\partial}_i, \gamma_i\})|_U = 0$. This implies that $\operatorname{im}\left[\{\overline{\partial}_i, \gamma_i\} - \operatorname{Id}\right]\Big|_U$ **lies in the space $R_i(U)$ of forms without $d\overline{z}_i$ in monomial decomposition and with all coefficients holomorphic as functions on z_i.**

Poincaré-Dolbeault-Grothendieck lemma (2)

THEOREM: Let $U \subset T^{2n}$ be a sufficiently small polydisk, and $\mu \in \Lambda^{p,q}(T^{2n})$ a form with $q > 0$ which is $\overline{\partial}$-closed on U. **Then there exists $\alpha \in \Lambda^{p,q-1}(T^{2n})$ such that $\overline{\partial}\alpha = \mu$ on U.**

Proof. Step 1: Let $\overline{\partial}_i : \Lambda^{p,q}(D^n) \longrightarrow \Lambda^{p,q+1}(D^n)$ be the operator $\alpha \longrightarrow d\overline{z}_i \wedge \frac{d}{d\overline{z}_i}\alpha$, and γ_i the homotopy defined above. Then $\operatorname{im}\left[\{\overline{\partial}_i, \gamma_i\} - \operatorname{Id}\right]\big|_U$ lies in the space $R_i(U)$ of forms without $d\overline{z}_i$ in monomial decomposition and with all coefficients holomorphic as functions on z_i.

Step 2: Let R_i denote the space of forms α on T^{2n} such that $\alpha|_U$ belongs to the space $R_i(U)$ defined above. Properties of γ_i:
(1). $\operatorname{im}\left[\{\overline{\partial}_i, \gamma_i\} - \operatorname{Id}\right] \subset R_i$. **(2).** $\{\overline{\partial}_i, \gamma_j\}|_U = 0$, **if** $i \neq j$. **(3). the restriction** $\left[\{\overline{\partial}_i, \gamma_i\}\right]\Big|_{R_i}$ **vanishes on** U. **(4).** $\gamma_i(R_j) \subset R_j$, $\overline{\partial}_i(R_j) \subset R_j$ **for all** $i \neq j$.
Property (1) is proven in Step 1, property (2) and (4) follow because γ_i is independent from the z_j-coordinate for all $j \neq i$. Finally, (3) follows because for all forms α without $d\overline{z}_i$ in its monomial decomposition one has $\{\gamma_i, \overline{\partial}\}(\alpha) = \gamma_i(\overline{\partial}_i(\alpha))$.

Step 3: Properties (1), (3) and (4) give $\left[\{\overline{\partial}_i, \gamma_i\} - \operatorname{Id}\right](R_{i_1} \cap R_{i_2} \cap ... \cap R_{i_k}) \subset R_i \cap R_{i_1} \cap R_{i_2} \cap ... \cap R_{i_k}$ for $i \notin \{i_1, i_2, ..., i_k\}$, and $\{\overline{\partial}_i, \gamma_i\}\Big|_{R_{i_1} \cap R_{i_2} \cap ... \cap R_{i_k}} = 0$ otherwise.

Poincaré-Dolbeault-Grothendieck lemma (3)

Step 3: Properties (1), (3) and (4) give $\left[\{\overline{\partial}_i, \gamma_i\} - \operatorname{Id}\right](R_{i_1} \cap R_{i_2} \cap ... \cap R_{i_k}) \subset R_i \cap R_{i_1} \cap R_{i_2} \cap ... \cap R_{i_k}$ for $i \notin \{i_1, i_2, ..., i_k\}$, and $\{\overline{\partial}_i, \gamma_i\}\Big|_{R_{i_1} \cap R_{i_2} \cap ... \cap R_{i_k}} = 0$ otherwise.

Step 4: Let $\gamma := \sum_i \gamma_i$. Since $\{\overline{\partial}_i, \gamma_j\} = 0$ for $i \neq j$, Step 3 gives

$$\left[\{\overline{\partial}, \gamma\} - (n-k)\operatorname{Id}\right](R_{i_1} \cap R_{i_2} \cap ... \cap R_{i_k}) \subset \sum_{i \neq i_1, i_2, ..., i_k} R_i \cap R_{i_1} \cap R_{i_2} \cap ... \cap R_{i_k}$$

Step 5: Let $W_0 = \Lambda^*(T^{2n})$, and $W_k \subset W_{k-1}$ the subspace generated by all $R_{i_1} \cap R_{i_2} \cap ... \cap R_{i_k}$ for $i_1 < i_2 < ... < i_k$. **Step 4 implies $\left[\{\overline{\partial}, \gamma\} - (n-k)\operatorname{Id}\right]\Big|_{W_k} \subset W_{k+1}$.**

Step 6: W_n is the space of $(p,0)$-forms holomorphic on U, and it does not contain any (p,q)-forms for $q > 0$. Using induction in $d = n-k$, **we can assume that any $\overline{\partial}$-closed (p,q)-form in W_{k+1} is $\overline{\partial}$-exact when $q > 0$. To prove PDG-lemma, it would suffice to prove the same for any $\overline{\partial}$-closed form $\alpha \in W_k$.** Step 5 gives $(n-k)\alpha - \{\overline{\partial}, \gamma\}(\alpha) = (n-k)\alpha - \overline{\partial}\gamma(\alpha) \in W_{k+1}$, and this form is $\overline{\partial}$-exact by the induction assumption. **This gives $(n-k)\alpha - \overline{\partial}\gamma(\alpha) = \overline{\partial}\eta$**, hence α is $\overline{\partial}$-exact. ■

Hartogs theorem

THEOREM: Let f be a holomorphic function on $\mathbb{C}^n \backslash K$, where $K \subset \mathbb{C}^n$ is a compact, and $n > 1$. **Then f can be extended to a holomorphic function on $\mathbb{C}^n$.**

Proof. Step 1: Replacing K by a bigger compact, we can assume that f is smoothly extended to a small neighbourhood of the closure $\overline{M\backslash K}$. Then f can be extended to a smooth function on $\mathbb{C}^n$, holomorphic outside of K. **Then $\alpha := \overline{\partial}\tilde{f}$ is a $\overline{\partial}$-closed $(0,1)$-form with compact support**.

Step 2: Using the standard open embedding of $\mathbb{C}^n$ to $\mathbb{C}P^n$, we may consider α as a $\overline{\partial}$-closed $(0,1)$-form on $\mathbb{C}P^n$. Since $H^1(\mathbb{C}P^n) = 0$, this gives $\alpha = \overline{\partial}\varphi$, where φ is a continuous function on $\mathbb{C}P^n$. In particular, **φ is bounded on $\mathbb{C}^n \subset \mathbb{C}P^n$**.

Step 3: Since $\overline{\partial}\varphi$ vanishes outside of K, the function φ is holomorphic outside of K. Since bounded holomorphic functions on $\mathbb{C}$ are constant, **φ is constant on any affine line not intersecting K**.

Step 4: This implies that $\varphi = const$ on the union of all affine lines not intersecting K. Since $n > 1$, the complement of this set is compact. Substracting constant if necessary, we obtain that **φ is a function with compact support**.

Step 5: $\overline{\partial}(\tilde{f} - \varphi) = \alpha - \alpha = 0$, **hence $\tilde{f} - \varphi$ is holomorphic**. However, φ has compact support, and therefore $f = \tilde{f} - \varphi$ outside of a compact. ■