Hodge theory, home assignment 1: complex structures on vector spaces

Rules: This is a class assignment for this week. Please bring your solutions (written) next Monday, or tell me your solutions during the week. We will have the class discussion the Wednesday after.

Exercise 1.1. Prove that the group $GL(n,\mathbb{C})$ of complex linear automorphisms of $\mathbb{C}^n$ is connected.

Definition 1.1. Let V be a vector space. A **complex structure** on V is an operator $I \in \operatorname{End}(V)$ which satisfies $I^2 = -\mathsf{Id}$.

Exercise 1.2. Let V be a real 4-dimensional vector space, equipped with a Euclidean metric, and S the space of all orthogonal complex structures $I \in \operatorname{End}(V)$. Prove that S is a disconnected union of two 2-dimensional spheres.

Exercise 1.3. Let V be an even-dimensional real vector space, and S the space of all complex structures on V. Prove that S has 2 connected components.

Exercise 1.4. Let V be an even-dimensional real vector space, equipped with a Euclidean metric g and S_g the space of all orthogonal complex structures on V. Prove that S has 2 connected components.

Exercise 1.5. Let V be a 4-dimensional real vector space, and U the space of antisymmetric 2-forms Ω on $V \otimes_{\mathbb{R}} \mathbb{C}$ such that $\Omega\wedge\Omega = 0$ and $\Omega \wedge \bar\Omega \neq 0$. Denote by $\mathbb{P}U$ the projectivization of U, that is, the quotient $U\backslash 0/\mathbb{C}^*$, under the standard $\mathbb{C}^*$-action. Construct a $GL(V)$-invariant bijective correspondence between $\mathbb{P}U$ and the space of complex structures on V.

Definition 1.2. Let g_1, g_2 be Riemannian metrics on a smooth manifold M. They are said to be **conformal**, or **conformally equivalent** if there exists a smooth function $\lambda \in C^\infty M$ such that $g_1 = \lambda \cdot g_2$. **Conformal structure** is a metric up to conformal equivalence.

Exercise 1.6. Let $f: M \longrightarrow N$ be an oriented diffeomorphism of almost complex Hermitian manifolds of real dimension 2. Prove that f is holomorphic if and only if it preserves the conformal structure.

Exercise 1.7 (*). Prove that there exists an oriented even-dimensional manifold not admitting an almost complex structure.