Hodge theory, home assignment 2: Frobenius form

Rules: This is a class assignment for this week. Please bring your solutions (written) next Monday, or tell me your solutions during the week. We will have the class discussion the Wednesday after.

Exercise 2.1. Find a vector field v on a 2-dimensional torus T^2 such that all orbits of the corresponding diffeomorphism flow are dense.

Exercise 2.2. Construct a 4-manifold M and a rank 2 distribution $B \subset TM$ such that $[B, B]$ has rank 3 and $[[B, B], B]$ has rank 4.

Definition 2.1. **A contact structure** on an $2n+1$-dimensional manifold M is a rank $2n$ distribution $B \subset TM$ such that TM/B is a trivial rank one bundle and the Frobenius form $\Lambda^2 B \longrightarrow TM/B$ is non-degenerate.

Exercise 2.3. Let θ be a 1-form on an $2n+1$-dimensional manifold M such that $\theta \wedge (d\theta)^n$ is non-degenerate, $f \in C^\infty M$ a nowhere vanishing function, and $\theta' = f\theta$. Prove that $\theta' \wedge (d\theta')^n$ is non-degenerate.

Exercise 2.4. Let θ be a 1-form on an $2n + 1$-dimensional manifold M such that $\theta \wedge (d\theta)^n$ is non-degenerate. Prove that $\ker\theta \subset TM$ is a contact distribution.

Exercise 2.5. Let M be an odd-dimensional manifold, and $B \subset TM$ a contact distribution. Prove that there exists a 1-form θ such that $B = \ker\theta$ and $\theta \wedge (d\theta)^n$ is non-degenerate.

Exercise 2.6. Construct a contact structure on a sphere S^{2n+1} for any $n = 1, 2, 3, ...$