Hodge theory, home assignment 3: almost complex structures and integrability

Rules: This is a class assignment for this week. Please bring your solutions (written) next Monday, or tell me your solutions during the week. We will have the class discussion the Wednesday after.

Definition 3.1. Let M be an almost complex manifold, and $A: \Lambda^*M \longrightarrow \Lambda^*M$ a linear map. **Hodge components** of A are operators $A^{p,q}$ such that $A = \sum_{p,q} A^{p,q}$ and $A^{p,q}(\Lambda^{i,j}(M)) \subset \Lambda^{i+p,j+q}(M)$.

Exercise 3.1. Prove that the de Rham differential on an almost complex manifold has at most 4 non-zero Hodge components: $d = d^{2,-1} + d^{1,0} + d^{0,1} + d^{-1,2}$.

Exercise 3.2. Let θ be a 1-form on a smooth manifold M, and $X, Y \in TM$ vector fields.

a. Let $i_X: \Lambda^i(M) \longrightarrow \Lambda^{i-1}(M)$ be the contraction map. Prove the Cartan's formula: $\operatorname{Lie}_X = i_x d + d i_X$.

b. Prove that $\operatorname{Lie}_X\langle\theta, Y\rangle = \theta([X,Y]) + \langle d(i_X\theta), Y\rangle + d\theta(X,Y)$

c. Prove that $\operatorname{Lie}_X\langle\theta, Y\rangle = -\theta([X,Y]) + d\theta(X,Y) + \operatorname{Lie}_Y\langle\theta, X\rangle$.

d. Prove another Cartan's formula: $d\theta(X,Y) = \theta([X,Y]) + \operatorname{Lie}_X\langle\theta, Y\rangle - - \operatorname{Lie}_Y\langle\theta, X\rangle$.

Hint. Use general properties of derivations to prove (a), deduce (b) from (a) and (c) from (b).

Exercise 3.3. Let (M, I) be an almost complex manifold, and $d = d^{2,-1} + d^{1,0} + d^{0,1} + d^{-1,2}$ the Hodge decomposition of the de Rham differential.

a. Prove that $d^{2,-1}$ and $d^{-1,2}$ are $C^\infty(M)$-linear.

b. Assume that $d^{2,-1} = d^{-1,2} = 0$. Prove that I is formally integrable.

c. Assume that $(d^{0,1})^2 = 0$. Prove that I is formally integrable.

Hint. For (b), use the Cartan's formula; deduce (c) from (b) and (d) from (c).

Exercise 3.4. Let (M, I) be an almost complex manifold.

a. Earlier, we denoted the weight (-1,2) Hodge component of de Rham differential by $d^{-1,2}$. Prove that $d^{-1,2}: \Lambda^{1,0}(M) \longrightarrow \Lambda^{0,2}(M)$ is dual to the complex conjugate of the Nijenhuis tensor $\bar{N}: \Lambda^2(T^{0,1}M) \longrightarrow T^{1,0}(M)$.

b. Prove that the operator $d^{-1,2}$ satisfies $d^{-1,2}(d\phi) = 0$ for any holomorphic function ϕ.

Exercise 3.5. Let (M, I) be an almost complex manifold. Assume that the Nijenhuis tensor $\Lambda^2(T^{1,0}M) \longrightarrow T^{0,1}(M)$ is surjective. Prove that (M, I) admits no non-constant local holomorphic functions.

Hint. Use the previous exercise.

Definition 3.2. Let G be a real Lie group, and $\mathfrak{g} = T_eG$ its Lie algebra. **The left action** of G on itself is a map $L_g(x) = gx$, defined for any $g \in G$. An almost complex structure is called **left invariant** if it is preserved by L_g for all $g \in G$.

Exercise 3.6. Let G be a real Lie group, and $\mathfrak{g} = T_eG$ its Lie algebra, and $\mathfrak{g}_{\mathbb{C}} := \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$ its complexification.

a. Prove that any complex structure operator on $\mathfrak{g} = T_eG$ is uniquely extended to a left-invariant almost complex structure on G.

b. Let $I \in \operatorname{End} T_eG$ be a complex structure, and $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g}^{1,0} \oplus \mathfrak{g}^{0,1}$ the Hodge decomposition of its complexification. Prove that I defines a formally integrable left-invariant complex structure on G if and only if $\mathfrak{g}^{1,0} \subset \mathfrak{g}_{\mathbb{C}}$ is a Lie subalgebra.

Exercise 3.7 (*). Construct a left-invariant almost complex structure I on a Lie group G such that (G, I) admits no local holomorphic functions.

Hint. Use Exercise 3.5.

Exercise 3.8. Let (M, I) be an almost complex manifold. Assume that the bundle $\Lambda^{1,0}(M)$ is generated by differentials of holomorphic functions. Without using the Newlander-Nirenberg theorem prove that the almost complex structure on M is integrable.