

Hodge theory

lecture 1: complex manifolds

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Erich Kähler (1906-2000)



Erich Kähler: "I have found the golden thread leading into the palpitating heart of mathematics".

Complex structure on a vector space

DEFINITION: Let V be a vector space over $\mathbb{R}$, and $I : V \rightarrow V$ an automorphism which satisfies $I^2 = -\text{Id}_V$. Such an automorphism is called **a complex structure operator** on V .

We extend the action of I on the tensor spaces $V \otimes V \otimes \dots \otimes V \otimes V^* \otimes V^* \otimes \dots \otimes V^*$ by multiplicativity: $I(v_1 \otimes \dots \otimes w_1 \otimes \dots \otimes w_n) = I(v_1) \otimes \dots \otimes I(w_1) \otimes \dots \otimes I(w_n)$.

Trivial observations:

1. **The eigenvalues α_i of I are $\pm\sqrt{-1}$.** Indeed, $\alpha_i^2 = -1$.
2. **V admits an I -invariant, positive definite scalar product (“metric”)**
 g . Take any metric g_0 , and let $g := g_0 + I(g_0)$.
3. **I is orthogonal for such g .**
 Indeed, $g(Ix, Iy) = g_0(x, y) + g_0(Ix, Iy) = g(x, y)$.
4. **I diagonalizable over $\mathbb{C}$.** Indeed, any orthogonal matrix is diagonalizable.
5. **There are as many $\sqrt{-1}$ -eigenvalues as there are $-\sqrt{-1}$ -eigenvalues.**

Complex structure operator in coordinates

This implies that in an appropriate basis in $V \otimes_{\mathbb{R}} \mathbb{C}$, the complex structure operator is diagonal, as follows:

$$\left[\begin{array}{cccc|cccc} \sqrt{-1} & & & & & & & \\ & \sqrt{-1} & & & & & & \\ & & \ddots & & & & & \\ & & & \sqrt{-1} & & & & \\ \hline & & & & -\sqrt{-1} & & & \\ & 0 & & & & -\sqrt{-1} & & \\ & & & & & & \ddots & \\ & & & & & & & -\sqrt{-1} \end{array} \right]$$

We also obtain its normal form in a real basis:

$$\left[\begin{array}{cc|cc|cc|cc} 0 & -1 & & & & & & \\ 1 & 0 & & & & & & \\ & & 0 & -1 & & & & \\ & & 1 & 0 & & & & \\ & & & & \ddots & & & \\ & & & & & \ddots & & \\ & & & & & & 0 & -1 \\ & & & & & & 1 & 0 \end{array} \right]$$

Hodge decomposition

DEFINITION: Let (V, I) be a space equipped with a complex structure.

The Hodge decomposition $V \otimes_{\mathbb{R}} \mathbb{C} := V^{1,0} \oplus V^{0,1}$ is defined in such a way that $V^{1,0}$ is a $\sqrt{-1}$ -eigenspace of I , and $V^{0,1}$ a $-\sqrt{-1}$ -eigenspace.

REMARK: In the same way one defines the Hodge decomposition on the dual space V^* .

Remark 1: The space $V^{1,0} \subset V \otimes_{\mathbb{R}} \mathbb{C}$ **uniquely determines the operator I** . Indeed, $I = \sqrt{-1}$ on $V^{1,0}$ and $I = -\sqrt{-1}$ on $V^{0,1}$. This gives **a bijection between the set of complex structures on V and the set of subspaces $W \subset V \otimes_{\mathbb{R}} \mathbb{C}$ such that $\dim_{\mathbb{C}} W = \frac{1}{2} \dim_{\mathbb{R}} V$ and $W \cap \overline{W} = 0$.**

Hermitian structures

DEFINITION: Let (V, I) be a space equipped with a complex structure. **The Hodge decomposition** $V \otimes_{\mathbb{R}} \mathbb{C} := V^{1,0} \oplus V^{0,1}$ is defined in such a way that $V^{1,0}$ is a $\sqrt{-1}$ -eigenspace of I , and $V^{0,1}$ a $-\sqrt{-1}$ -eigenspace.

DEFINITION: An I -invariant positive definite scalar product on (V, I) is called **an Hermitian metric**, and (V, I, g) – an Hermitian space.

REMARK: Let I be a complex structure operator on a real vector space V , and g – a Hermitian metric. Then **the bilinear form** $\omega(x, y) := g(x, Iy)$ **is skew-symmetric**. Indeed, $\omega(x, y) = g(x, Iy) = g(Ix, I^2y) = -g(Ix, y) = -\omega(y, x)$.

DEFINITION: A skew-symmetric form $\omega(x, y)$ is called **an Hermitian form on (V, I)** .

REMARK: In the triple I, g, ω , **each element can recovered from the other two**.

Almost complex structures

DEFINITION: Let $I : TM \rightarrow TM$ be an endomorphism of a tangent bundle satisfying $I^2 = -\text{Id}$. Then I is called **almost complex structure operator**, and the pair (M, I) **an almost complex manifold**.

EXAMPLE: $M = \mathbb{C}^n$, with complex coordinates $z_i = x_i + \sqrt{-1} y_i$, and $I(d/dx_i) = d/dy_i$, $I(d/dy_i) = -d/dx_i$.

EXAMPLE: In complex dimension 1, **almost complex structure is the same as conformal structure with orientation (prove it)**.

DEFINITION: A function $f : M \rightarrow \mathbb{C}$ on an almost complex manifold is called **holomorphic** if $df \in \Lambda^{1,0}(M)$.

REMARK: For some almost complex manifolds, **there are no holomorphic functions at all**, even locally.

Example: S^6 with the unique G_2 -invariant almost complex structure.

Holomorphic functions on $\mathbb{C}^n$

THEOREM: Let $f : M \rightarrow \mathbb{C}$ be a differentiable function on an open subset $M \subset \mathbb{C}^n$, with almost complex structure as above. **Then TFAE:**

- (1) f is holomorphic.
- (2) The differential $df : TM \rightarrow \mathbb{C}$, considered as a form on the vector space $T_x M = T_x \mathbb{C}^n = \mathbb{C}^n$ is $\mathbb{C}$ -linear.
- (3) For any complex affine line $L \subset \mathbb{C}^n$, the restriction $f|_L : L \rightarrow \mathbb{C}$ is holomorphic (complex analytic) as a function of one complex variable.
- (4) f is expressed as a sum of Taylor series around any point $(z_1, \dots, z_n) \in M$.

Proof: (1) and (2) are tautologically equivalent. Equivalence of (1) and (3) is also clear, because a restriction of $\theta \in \Lambda^{1,0}(M)$ to a line is a $(1,0)$ -form on a line, and, conversely, if df is of type $(1,0)$ on each complex line, it is of type $(1,0)$ on TM , which is implied by the following linear-algebraic observation.

LEMMA: Let $\eta \in V^* \otimes \mathbb{C}$ be a complex-valued linear form on a vector space (V, I) equipped with a complex structure. **Then $\eta \in \Lambda^{1,0}(V)$ if and only if its restriction to any I -invariant 2-dimensional subspace L belongs to $\Lambda^{1,0}(L)$.**

EXERCISE: Prove it.

(4) clearly implies (2). (1) implies (4) by Cauchy formula.

Sheaves

DEFINITION: A **presheaf of functions** on a topological space M is a collection of subrings $\mathcal{F}(U) \subset C(U)$ in the ring $C(U)$ of all functions on U , for each open subset $U \subset M$, such that the restriction of every $\gamma \in \mathcal{F}(U)$ to an open subset $U_1 \subset U$ belongs to $\mathcal{F}(U_1)$.

DEFINITION: A presheaf of functions $\mathcal{F}$ is called **a sheaf of functions** if these subrings satisfy the following condition. Let $\{U_i\}$ be a cover of an open subset $U \subset M$ (possibly infinite), and f a function on U such that $f|_{U_i}$ belongs to $\mathcal{F}(U_i)$. Then $f \in \mathcal{F}(U)$.

Sheaves and presheaves: examples

Examples of sheaves:

- * Space of continuous functions
- * Space of smooth functions, any differentiability class
- * Space of real analytic functions

Examples of presheaves which are not sheaves:

- * Space of constant functions (why?)
- * Space of bounded functions (why?)

Ringed spaces

A **ringed space** $(M, \mathcal{F})$ is a topological space equipped with a sheaf of functions. A **morphism** $(M, \mathcal{F}) \xrightarrow{\Psi} (N, \mathcal{F}')$ of ringed spaces is a continuous map $M \xrightarrow{\Psi} N$ such that, for every open subset $U \subset N$ and every function $f \in \mathcal{F}'(U)$, the function $\psi^* f := f \circ \Psi$ belongs to the ring $\mathcal{F}(\Psi^{-1}(U))$. An **isomorphism** of ringed spaces is a homeomorphism Ψ such that Ψ and Ψ^{-1} are morphisms of ringed spaces.

EXAMPLE: Let M be a manifold of class C^i and let $C^i(U)$ be the space of functions of this class. **Then C^i is a sheaf of functions, and (M, C^i) is a ringed space.**

REMARK: Let $f : X \rightarrow Y$ be a smooth map of smooth manifolds. Since a pullback $f^* \mu$ of a smooth function $\mu \in C^\infty(M)$ is smooth, **a smooth map of smooth manifolds defines a morphism of ringed spaces.**

Converse is also true:

Ringed spaces and smooth maps

CLAIM: Let (M, C^i) and (N, C^i) be manifolds of class C^i . Then **there is a bijection between smooth maps $f : M \rightarrow N$ and the morphisms of corresponding ringed spaces.**

Proof: Any smooth map induces a morphism of ringed spaces. Indeed, **a composition of smooth functions is smooth, hence a pullback is also smooth.**

Conversely, let $U_i \rightarrow V_i$ be a restriction of f to some charts; to show that f is smooth, it would suffice to show that $U_i \rightarrow V_i$ is smooth. However, we know that a pullback of any smooth function is smooth. **Therefore, Claim is implied by the following lemma.**

LEMMA: Let M, N be open subsets in $\mathbb{R}^n$ and let $f : M \rightarrow N$ map such that a pullback of any function of class C^i belongs to C^i . **Then f is of class C^i .**

Proof: Apply f to coordinate functions. ■

Smooth manifolds defined through sheaves

As we have just shown, this definition is equivalent to the previous one.

DEFINITION: Let $(M, \mathcal{F})$ be a topological manifold equipped with a sheaf of functions. It is said to be a **smooth manifold of class** C^∞ or C^i if every point in $(M, \mathcal{F})$ has an open neighborhood isomorphic to the ringed space $(\mathbb{R}^n, \mathcal{F}')$, where $\mathcal{F}'$ is a ring of functions on $\mathbb{R}^n$ of this class.

DEFINITION: A **chart**, or a **coordinate system** on an open subset U of a manifold $(M, \mathcal{F})$ is an isomorphism between $(U, \mathcal{F})$ and an open subset in $(\mathbb{R}^n, \mathcal{F}')$, where $\mathcal{F}'$ are functions of the same class on $\mathbb{R}^n$.

DEFINITION: **Diffeomorphism** of smooth manifolds is a homeomorphism φ which induces an isomorphism of ringed spaces, that is, φ and φ^{-1} map (locally defined) smooth functions to smooth functions.

Complex manifolds

DEFINITION: A holomorphic function on $\mathbb{C}^n$ is a function $f : \mathbb{C}^n \rightarrow \mathbb{C}$ such that df is complex linear, that is $df \in \Lambda^{1,0}(M)$.

REMARK: Holomorphic functions form a sheaf.

DEFINITION: A complex manifold M is a ringed space which is locally isomorphic to an open ball in $\mathbb{C}^n$ with a sheaf of holomorphic functions.

REMARK: In other words, M is covered with open balls embedded to $\mathbb{C}^n$ and transition functions (being coordinate functions for one ball considered in other coordinate system) are holomorphic.

Complex manifolds and almost complex manifolds

DEFINITION: **Standard almost complex structure** is $I(d/dx_i) = d/dy_i$, $I(d/dy_i) = -d/dx_i$ on $\mathbb{C}^n$ with complex coordinates $z_i = x_i + \sqrt{-1} y_i$.

DEFINITION: A map $\psi : (M, I) \longrightarrow (N, J)$ from an almost complex manifold to an almost complex manifold is called **holomorphic** if $\psi^*(\Lambda^{1,0}(N)) \subset \Lambda^{1,0}(M)$.

REMARK: This is the same as $d\psi$ being complex linear; for standard almost complex structures, **this is the same as the coordinate components of ψ being holomorphic functions.**

Integrability of almost complex structures

DEFINITION: An almost complex structure I on a manifold is called **integrable** if any point of M has a neighbourhood U diffeomorphic to an open subset of $\mathbb{C}^n$, in such a way that the almost complex structure I is induced by the standard one on $U \subset \mathbb{C}^n$.

CLAIM: Let M be a manifold, and $I \in \text{End}(TM)$ an integrable almost complex structure. Denote by $\mathcal{O}_M$ the sheaf of holomorphic functions on M . Then the ringed space $(M, \mathcal{O}_M)$ **is a complex manifold**.

Proof: Clear from the definition. ■

CLAIM: **Complex structure on a manifold M uniquely determines an integrable almost complex structure, and is determined by it.**

Proof: Complex structure on a manifold M is determined by the sheaf of holomorphic functions $\mathcal{O}_M$, and $\mathcal{O}_M$ is determined by I as explained above. Therefore, an integrable almost complex structure defines a complex structure. Conversely, every complex structure gives a sub-bundle in $\Lambda^{1,0}(M) = d\mathcal{O}_M \subset \Lambda^1(M, \mathbb{C})$, and **such a sub-bundle defines an almost complex structure by Remark 1.** ■