Hodge theory

lecture 2: real analytic manifolds

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Real analytic manifolds

DEFINITION: A real analytic function on an open set $U \subset \mathbb{R}^n$ is a function which admits a Taylor expansion near each point $x \in U$:

$$f(z_1 + t_1, z_2 + t_2, ..., z_n + t_n) = \sum_{i_1,...,i_n} a_{i_1,...,i_n} t_1^{i_1} t_2^{i_2} ... t_n^{i_n}.$$

(here we assume that the real numbers t_i satisfy $|t_i| < \varepsilon$, where ε depends on f and M).

REMARK: Clearly, **real analytic functions constitute a sheaf.**

DEFINITION: A **real analytic manifold** is a ringed space which is locally isomorphic to an open ball $B \subset \mathbb{R}^n$ with the sheaf of real analytic functions.

REMARK: Today I will be giving **another definition of real analytic manifolds,** due to Grauert. It is equivalent to this one, but way better suited for complex analytic applications.

Involutions

DEFINITION: An involution is a map $\iota: M \longrightarrow M$ such that $\iota^2 = \operatorname{Id}_M$.

EXERCISE: Prove that **any linear involution on a real vector space V is diagonalizable,** with eigenvalues ± 1.

Theorem 1: Let M be a smooth manifold, and $\iota: M \longrightarrow M$ an involution. **Then the fixed point set N of ι is a smooth submanifold.**

Proof. Step 1: Inverse function theorem. Let $m \in M$ be a point on a smooth k-dimensional manifold and $f_1, ..., f_k$ functions on M such that their differentials $df_1, ..., df_k$ are linearly independent in m. Then $f_1, ..., f_k$ **define a coordinate system in a neighbourhood of a, giving a diffeomorphism of this neighbourhood to an open ball.**

Step 2: Assume that $d\iota$ has k eigenvalues 1 on $T_m M$, and $n-k$ eigenvalues -1. Choose a coordinate system $x_1, ..., x_n$ on M around a point $m \in N$ such that $dx_1|_m, ..., dx_k|_m$ are ι-invariant and $dx_{k+1}|_m, ..., dx_n|_m$ are ι-anti-invariant. Let $y_1 = x_1 + \iota^* x_1$, $y_2 = x_2 + \iota^* x_2$, ... $y_k = x_k + \iota^* x_k$, and $y_{k+1} = x_{k+1} - \iota^* x_{k+1}$, $y_{k+2} = x_{k+2} - \iota^* x_{k+2}$, ... $y_n = x_n - \iota^* x_n$. Since $dx_i|_m = dy_i|_m$, these differentials are linearly independent in m. By Step 1, **functions y_i define an ι-invariant coordinate system on an open neighbourhood of m, with N given by equations $y_{k+1} = y_{k+2} = ... = y_n = 0$.** ■

Real structures on complex vector spaces

DEFINITION: A real structure on a complex vector space $V = \mathbb{C}^n$ is an $\mathbb{R}$-linear involution $\iota : V \longrightarrow V$ such that $\iota(\lambda x) = \overline{\lambda}\iota(x)$ for any $\lambda \in \mathbb{C}$.

Claim 1: Let ι be a real structure on a complex vector space V, and $V^\iota \subset V$ the space of V-invariant vectors. **Then $\dim_{\mathbb{R}} V^\iota = \dim_{\mathbb{C}} V$, and $V = V^\iota \otimes_{\mathbb{R}} \mathbb{C}$.**

Proof. Step 1: Let $x_1, ..., x_n$ be a basis in V^ι, and $\sum_i \alpha_i x_i = 0$ a linear relation in V, with $\alpha_i \in \mathbb{C}$. Then $0 = \iota\left(\sum_i \alpha_i x_i\right) = \sum_i \overline{\alpha}_i x_i$. Averaging these two relations, we obtain $\sum \operatorname{Re} \alpha_i x_i = 0$. Since x_i are linearly independent over $\mathbb{R}$, this implies $\operatorname{Re} \alpha_i = 0$ for all i. Applying the same argument to $\sum_i \sqrt{-1}\, \alpha_i x_i = 0$, we obtain that $\operatorname{Im} \alpha_i = 0$ for all i. Then **the natural map $V^\iota \otimes_{\mathbb{R}} \mathbb{C} \longrightarrow V$ is injective.**

Step 2: This map is also surjective. Indeed, for any $v \in V$, one has $\frac{1}{2}(v + \iota(v)) \in V^\iota$ and $\frac{\sqrt{-1}}{2}(v - \iota(v)) \in V^\iota$, hence v can b expressed as a linear combination of vectors from V^ι with complex coefficients. ■

Real structures on complex vector spaces: an equivalence of categories

Let $\mathcal{C}_\iota$ the category whose objects are complex vector spaces equipped with an anticomplex involution and morphisms are complex linear maps $(V_1, \iota_1) \to (V_2, \iota_2)$ commuting with the real structures. Let $\mathcal{C}_{\mathbb{R}}$ be the category of real vector spaces. Consider the functor $\Psi: \mathcal{C}_\iota \to \mathcal{C}_{\mathbb{R}}$ taking (V, ι) to the fixed point set V^ι.

THEOREM: The functor $\Psi: \mathcal{C}_\iota \to \mathcal{C}_{\mathbb{R}}$ is a equivalence of categories.

Proof: Let $\Phi: \mathcal{C}_{\mathbb{R}} \to \mathcal{C}_\iota$ take a real vector space V to $V_{\mathbb{C}} := V \otimes_{\mathbb{R}} \mathbb{C}$ with the involution $x \otimes \lambda \to x \otimes \overline{\lambda}$. Clearly, $\Psi(\Phi(V)) = V$, and $\Phi(\Psi(W, \iota))$ takes (W, ι) to $(W^\iota \otimes_{\mathbb{R}} \mathbb{C})$, which is isomorphic to (W, ι) (Claim 1). ■

Real structures on complex manifolds

DEFINITION: A smooth map $\Psi: M \longrightarrow N$ on an almost complex manifold (M, I) is called **antiholomorphic** if $d\Psi(I) = -I$. A function f is called **antiholomorphic** if $\overline{f}$ is holomorphic.

EXERCISE: Prove that **an antiholomorphic function on M defines an antiholomorphic map from M to $\mathbb{C}$.**

EXERCISE: Prove that a map $\Psi: M \longrightarrow N$ of almost complex manifolds is antiholomorphic **if and only if $\Psi^*(\Lambda^{0,1}(N)) \subset \Lambda^{1,0}(M)$.**

EXERCISE: Let ι be a smooth map from a complex manifold M to itself. Prove that **ι is antiholomorphic if and only if $\iota^*(f)$ is antiholomorphic for any holomorphic function f on $U \subset M$.**

DEFINITION: A real structure on a complex manifold M is an antiholomorphic involution $\tau: M \longrightarrow M$.

Fixed points of real structures on manifolds

PROPOSITION: Let M be a complex manifold and $\iota: M \longrightarrow M$ a real structure. Denote by M^ι the fixed point set of ι. Then, **for each $x \in M^\iota$ there exists a ι-invariant coordinate neighbourhood with holomorphic coordinates $z_1, ..., z_n$, such that $\iota^*(z_i) = \overline{z}_i$.**

Proof. Step 1: For each basis of 1-forms $\nu_1, ..., \nu_n \in \Lambda^{1,0}_x(M)$, there exists a set of holomorphic coordinate functions $u_1, ..., u_n$ such that $du_i|_x = \nu_i$. To obtain such a coordinate system, **we chose any coordinate system $v_1, ..., v_n$ and apply a linear transform mapping $dv_i|_x$ to ν_i.**

Step 2: The differential $d\iota$ acts on T_xM as a real structure. Using the structure theorem about the real structures on complex vector spaces, we obtain that any real basis $\zeta_1, .., \zeta_n$ of $T^*_x M^\iota$ is a complex basis in the complex vector space $T^*_x M$. Then $\nu_i := \zeta_i + \sqrt{-1}\, I(\zeta_i)\}$ is a basis in $\Lambda^{1,0}_x(M)$. Choose the coordinate system $u_1, ..., u_n$ such that $du_i|_x = \nu_i$ (Step 1). **Replacing u_i by $z_i := u_i + \iota^*(\overline{u}_i)$, we obtain a holomorphic coordinate system z_i on M** (compare with Theorem 1 in Lecture 4) **which satisfies $\iota^*(z_i) = \overline{z}_i$.** ■

DEFINITION: Let $\{U_i\}$ be an complex atlas on M. Assume that any U_i intersecting M^ι satisfies the conclusion of this proposition. Then $\{U_i\}$ is called **an atlas compatible with the real structure**.

Real analytic manifolds and real structures

PROPOSITION: Let $M^\iota \subset M$ be a fixed point set of an antiholomorphic involution ι on a complex manifold M, $\{U_i\}$ a complex analytic atlas, and $\Psi_{ij}: U_{ij} \longrightarrow U_{ij}$ the gluing functions. Assume that the atlas U_i is compatible with the real structure, in the sense of the previous proposition. **Then all Ψ_{ij} are real analytic on M^ι, and define a real analytic atlas on the manifold M^ι.**

Proof: All gluing functions from one coordinate system compatible with the real structure to another **commute with ι, acting on coordinate functions as the complex conjugation.** This gives $\Psi_{ij}(\overline{z}_i) = \overline{\Psi_{ij}(z_i)}$. Therefore, Ψ_{ij} preserve M^ι, and are expressed by real-valued functions on M^ι. ■

Real analytic manifolds and real structures 2

PROPOSITION: Any real analytic manifold can be obtained from this construction.

Proof. Step 1: Let $\{U_i\}$ be a locally finite atlas of a real analytic manifold M, and $\Psi_{ij} : U_{ij} \longrightarrow U_{ij}$ the gluing maps. We realize U_i as an open ball with compact closure in $\operatorname{Re}(\mathbb{C}^n) = \mathbb{R}^n$. By local finiteness, there are only finitely many such Ψ_{ij} for any given U_i. Denote by B_ε an open ball of radius ε in the n-dimensional real space $\operatorname{im}(\mathbb{C}^n)$.

Step 2: Let $\varepsilon > 0$ be a sufficiently small real number such that all Ψ_{ij} can be extended to gluing functions $\tilde{\Psi}_{ij}$ on the open sets $\tilde{U}_i := U_i \times B_\varepsilon \subset \mathbb{C}^n$. **Then $(\tilde{U}_i, \Psi_{ij})$ is an atlas for a complex manifold $M_{\mathbb{C}}$.** Since all Ψ_{ij} are real, they are preserved by the natural involution acting on B_ε as -1 and on U_i as identity. This involution defines a real structure on $M_{\mathbb{C}}$. Clearly, M is the set of its fixed points. ■

Complexification

DEFINITION: Let $M_{\mathbb R}$ be a real analytic manifold, and $M_{\mathbb C}$ a complex analytic manifold equipped with an antiholomorphic involution, such that $M_{\mathbb R}$ is the set of its fixed points. Then $M_{\mathbb C}$ is called **a complexification** of $M_{\mathbb R}$.

EXAMPLE: Consider a complex torus $\mathbb C/\mathbb Z^2$ with the involution $z \to \overline{z}$. Clearly, fixed points of this involution is a circle $\mathbb R/\mathbb Z$. Therefore, **the torus is a complexification of S^1.**

EXAMPLE: Consider $\mathbb C^* := \mathbb C\backslash 0$ with the involution $z \to \overline{z}^{-1}$. Clearly, fixed point set of this involution is a circle $\{z \in \mathbb C^* \ \ | \ \ |z| = 1\}$. Therefore, **$\mathbb C^*$ is a complexification of S^1.**

EXAMPLE: Consider $\mathbb CP^1$ with the involution $x : y \mapsto \overline{x} : \overline{y}$. Clearly, the fixed point set of this involution is $\mathbb RP^1 = S^1$. Therefore, **$\mathbb CP^1$ is a complexification of S^1.**

DEFINITION: A tensor on a real analytic manifold is called **real analytic** if it is expressed locally by a sum of coordinate monomials with real analytic coefficients.

Germ of a complex manifold

DEFINITION: Let $K \subset M$ be a closed subset of a complex manifold, homeomorphic to $K_1 \subset M_1$, where M_1 is also a complex manifold. Fixing the homeomorphism $K \cong K_1$, we may identify these sets and consider K as a subset M_1. We say that M and M_1 **have the same germ in K** if there exist biholomorphic open subsets $U_1 \subset M_1$ and $U \subset M$ containing K, with the biholomorphism $\varphi : U \longrightarrow U_1$ identity on K.

DEFINITION: **Germ of a manifold M in $K \subset M$** is an equivalence class of open subsets $U \subset M$ containing K, with this equivalence relation.

DEFINITION: Consider category $\mathcal{C}_\iota$, with objects complex manifolds (M, ι) equipped with a real structure, and morphisms holomorphic maps commuting with ι.

THEOREM: **(Grauert) Category of real analytic manifolds is equivalent to the category of germs of $M \in \mathcal{C}_\iota$ in $M^\iota \subset M$.**

EXERCISE: **Prove this theorem.**

Hans Grauert

Hans Grauert in Bonn, 2000

(8.02.1930 - 4.09.2011)

Antiholomorphic functions

DEFINITION: A function f on a complex manifold (M,I) is called **antiholomorphic** if $df \in \Lambda^{0,1}(M)$. Clearly, $\overline{f}$ is holomorphic: complex conjugation defines an isomorphism between the sheaf of holomorphic and antiholomorphic functions.

REMARK: A function f on (M,I) **is holomorphic if and only if f is antiholomorphic on $(M,-I)$.**

DEFINITION: An antiholomorphic tensor on a complex manifold (M,I) is a holomorphic tensor on $(M,-I)$.

REMARK: Let ι be an anticomplex involution of a complex manifold M. **Then ι^* takes holomorphic function to antiholomorphic ones, and holomorphic tensors to antiholomorphic ones.**

Holomorphic tensors on the complexification

THEOREM: Let $M_{\mathbb{R}}$ be a real analytic manifold and $(M_{\mathbb{C}}, \iota)$ its complexification. Then each real analytic tensor $\psi_{\mathbb{R}}$ on $M_{\mathbb{R}}$ **can be extended to a holomorphic tensor $\psi_{\mathbb{C}}$ on $M_{\mathbb{C}}$,** defined in some neighbourhood of $M_{\mathbb{R}} = M_{\mathbb{C}}^{\iota}$ which satisfies $\iota^*(\psi_{\mathbb{C}}) = \overline{\psi}_{\mathbb{C}}$. Moreover, **this correspondence defines an equivalence:** every holomorphic tensor in a neighbourhood of $M_{\mathbb{R}} \subset M_{\mathbb{C}}$ which satisfies $\iota^*(\psi_{\mathbb{C}}) = \overline{\psi}_{\mathbb{C}}$ is obtained this way.

Proof. Step 1: Every complex analytic tensor ψ on $M_{\mathbb{C}}$ is by definition real analytic on $M_{\mathbb{R}}$. If ψ satisfies $\iota^*(\psi_{\mathbb{C}}) = \overline{\psi}_{\mathbb{C}}$, then its restriction to $M_{\mathbb{R}}$ is real. **Indeed, $\psi_{\mathbb{C}}\big|_{M_{\mathbb{R}}}$ satisfies $\overline{\psi}_{\mathbb{C}}\big|_{M_{\mathbb{R}}} = \psi_{\mathbb{C}}\big|_{M_{\mathbb{R}}}$.**

Step 2: Conversely, suppose that a real analytic tensor φ is expressed in coordinates by a sum of tensorial monomials with real analytic coefficients f_i. Let $\{U_i\}$ be a cover of M, and $\tilde{U}_i := U_i \times B_\varepsilon$ the corresponding cover of a neighbourhood of $M_{\mathbb{R}}$ in $M_{\mathbb{C}}$. Choosing ε sufficiently small, we can assume that the Taylor series giving coefficients of φ converges on each $\tilde{U}_i$. **We define $\varphi_{\mathbb{C}}$ as the sum of these series;** it converges on a sufficiently small neighbourhood of $M_{\mathbb{R}}$. ■