Hodge theory

lecture 3: Newlander-Nirenberg theorem

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Real analytic manifolds (reminder)

DEFINITION: A real analytic function on an open set $U \subset \mathbb{R}^n$ is a function which admits a Taylor expansion near each point $x \in U$:

$$f(z_1 + t_1, z_2 + t_2, ..., z_n + t_n) = \sum_{i_1,...,i_n} a_{i_1,...,i_n} t_1^{i_1} t_2^{i_2} ... t_n^{i_n}.$$

(here we assume that the real numbers t_i satisfy $|t_i| < \varepsilon$, where ε depends on f and M).

REMARK: Clearly, **real analytic functions constitute a sheaf.**

DEFINITION: A **real analytic manifold** is a ringed space which is locally isomorphic to an open ball $B \subset \mathbb{R}^n$ with the sheaf of real analytic functions.

Real structures on complex manifolds (reminder)

DEFINITION: A smooth map $\Psi: M \longrightarrow N$ on an almost complex manifold (M, I) is called **antiholomorphic** if $d\Psi(I) = -I$. A function f is called **antiholomorphic** if $\overline{f}$ is holomorphic.

EXERCISE: Prove that **an antiholomorphic function on M defines an antiholomorphic map from M to $\mathbb{C}$.**

EXERCISE: Prove that a map $\Psi: M \longrightarrow N$ of almost complex manifolds is antiholomorphic **if and only if $\Psi^*(\Lambda^{0,1}(N)) \subset \Lambda^{1,0}(M)$.**

EXERCISE: Let ι be a smooth map from a complex manifold M to itself. Prove that **ι is antiholomorphic if and only if $\iota^*(f)$ is antiholomorphic for any holomorphic function f on $U \subset M$.**

DEFINITION: A real structure on a complex manifold M is an antiholomorphic involution $\tau: M \longrightarrow M$.

Fixed points of real structures on complex manifolds (reminder)

PROPOSITION: Let M be a complex manifold and $\iota: M \longrightarrow M$ a real structure. Denote by M^ι the fixed point set of ι. Then, **for each $x \in M^\iota$ there exists a ι-invariant coordinate neighbourhood with holomorphic coordinates $z_1, ..., z_n$, such that $\iota^*(z_i) = \overline{z}_i$.**

Proof: Lecture 2. ■

PROPOSITION: **Any real analytic manifold can be obtained from this construction.**

Proof: Lecture 2. ■

Complexification

DEFINITION: Let $M_{\mathbb R}$ be a real analytic manifold, and $M_{\mathbb C}$ a complex analytic manifold equipped with an antiholomorphic involution, such that $M_{\mathbb R}$ is the set of its fixed points. Then $M_{\mathbb C}$ is called **a complexification** of $M_{\mathbb R}$.

DEFINITION: A tensor on a real analytic manifold is called **real analytic** if it is expressed locally by a sum of coordinate monomials with real analytic coefficients.

CLAIM: Let $M_{\mathbb R}$ be a real analytic manifold, $(M_{\mathbb C}, \iota)$ its complexification, and Φ a tensor on $M_{\mathbb R}$. **Then Φ is real analytic if and only if Φ can be extended to a holomorpic tensor $\Phi_{\mathbb C}$ in some neighbourhood of $M_{\mathbb R}$ inside $M_{\mathbb C}$.** Moreover, **Φ is real on $M_{\mathbb R}$ if $\iota^*\Phi_{\mathbb C} = \overline{\Phi}_{\mathbb C}$.**

Proof: The "if" part is clear, because every complex analytic tensor on $M_{\mathbb C}$ is by definition real analytic on $M_{\mathbb R}$.

Conversely, suppose that Φ is expressed in coordinates by a sum of tensorial monomials with real analytic coefficients f_i. Let $\{U_i\}$ be a cover of M, and $\tilde U_i := U_i \times B_\varepsilon$ the corresponding cover of a neighbourhood of $M_{\mathbb R}$ in $M_{\mathbb C}$ constructed in Lecture 4. Chosing ε sufficiently small, we can assume that the Taylor series giving coefficients of Φ converges on each $\tilde U_i$. **We define $\Phi_{\mathbb C}$ as the sum of these series.** ■

Extension of tensors to a complexification

Lemma 1: Let X be an open ball in $\mathbb{C}^n$ equipped with the standard anticomplex involution, $X_{\mathbb{R}} = X \cap \mathbb{R}^n$ its fixed point set, and α a holomorphic tensor on X vanishing in $X_{\mathbb{R}}$. **Then $\alpha = 0$.**

Proof: Any holomorphic function which vanishes on $\mathbb{R}^n$ has all its derivatives vanishing. Therefore its Taylor series vanish. Such a function vanishes on $\mathbb{C}^n$ by analytic continuation principle. This argument can be applied to all coefficients of α. ■

DEFINITION: An almost complex structure I on a real analytic manifold is **real analytic** if I is a real analytic tensor.

COROLLARY: Let (M, I) be a real analytic almost complex manifold, $M_{\mathbb{C}}$ its complexification, and $I_{\mathbb{C}}: TM_{\mathbb{C}} \longrightarrow TM_{\mathbb{C}}$ the holomorphic extension of I to $M_{\mathbb{C}}$. **Then $I_{\mathbb{C}}^2 = -\operatorname{Id}$.**

Proof: The tensor $I_{\mathbb{C}}^2 + \operatorname{Id}$ is holomorphic and vanishes on $M_{\mathbb{R}}$, hence the previous lemma can be applied. ■

Underlying real analytic manifold

REMARK: A complex analytic map $\Phi: \mathbb{C}^n \longrightarrow \mathbb{C}^n$ is real analytic as a map $\mathbb{R}^{2n} \longrightarrow \mathbb{R}^{2n}$. Indeed, the coefficients of Φ are real and imaginary parts of holomorphic functions, and real and imaginary parts of holomorphic functions can be expressed as Taylor series of the real variables.

DEFINITION: Let M be a complex manifold. The **underlying real analytic manifold** $M_{\mathbb{R}}$ is the same manifold, with the same gluing functions, considered as real analytic maps.

REMARK: The sheaf of real analytic functions on $M_{\mathbb{R}}$ can be defined as **the sheaf of converging power series generated by holomorphic and antiholomorphic functions.** Indeed, such functions are real analytic in any of the real analytic map; conversely, **any real analytic function on $M_{\mathbb{R}}$ is a converging power series on $\operatorname{Re} z_i, \operatorname{Im} z_i$, where z_i are holomorphic coordinates on M.**

Complexification of the underlying real analytic manifold

DEFINITION: Let M be a complex manifold. The **complex conjugate manifold** is the same manifold with almost complex structure $-I$ and anti-holomorphic functions on M holomorphic on $\overline{M}$.

CLAIM: Let M be an integrable almost complex manifold. Denote by $M_{\mathbb{R}}$ its underlying real analytic manifold. **Then a complexification of $M_{\mathbb{R}}$ can be given as $M_{\mathbb{C}} := M \times \overline{M}$,** with the anticomplex involution $\tau(x, y) = (y, x)$.

Proof: Clearly, the fixed point set of τ is the diagonal, identified with $M_{\mathbb{R}} = M$ as usual. Both holomorphic and antiholomorphic functions on $M_{\mathbb{R}}$ are obtained as restrictions of holomorphic functions from $M_{\mathbb{C}}$, hence the sheaf of real analytic functions on $M_{\mathbb{R}}$ is a subsheaf of $\mathcal{O}_{M_{\mathbb{C}}}$ of holomorphic functions on $M_{\mathbb{C}}$ restricted to $M_{\mathbb{R}}$. ∎

Integrability of almost complex structures (reminder)

DEFINITION: An almost complex structure I on a manifold is called **integrable** if any point of M has a neighbourhood U diffeomorphic to an open subset of $\mathbb{C}^n$, in such a way that the almost complex structure I is induced by the standard one on $U \subset \mathbb{C}^n$.

CLAIM: Let M be a manifold, and $I \in \operatorname{End}(TM)$ an integrable almost complex structure. Denote by $\mathcal{O}_M$ the sheaf of holomorphic functions on M. Then the ringed space $(M, \mathcal{O}_M)$ **is a complex manifold**.

Proof: Clear from the definition. ■

CLAIM: Complex structure on a manifold M uniquely determines an integrable almost complex structure, and is determined by it.

Proof: Complex structure on a manifold M is determined by the sheaf of holomorphic functions $\mathcal{O}_M$, and $\mathcal{O}_M$ is determined by I as explained above. Therefore, an integrable almost complex structure defines a complex structure. Conversely, every complex structure gives a sub-bundle in $\Lambda^{1,0}(M) = d\mathcal{O}_M \subset \Lambda^1(M, \mathbb{C})$, and **such a sub-bundle defines an almost complex structure by Remark 1.** ■

Formal integrability

DEFINITION: Let $I: TM \longrightarrow TM$ be an almost complex structure on M, and $TM \otimes_{\mathbb{R}} \mathbb{C} = T^{1,0}M \oplus T^{0,1}$ the Hodge decomposition. An almost complex structure I on (M, I) is called **formally integrable** if $[T^{1,0}M, T^{1,0}] \subset T^{1,0}$, that is, if $T^{1,0}M$ is involutive.

DEFINITION: The Frobenius form $\Psi \in \Lambda^{2,0}M \otimes TM$ is called **the Nijenhuis tensor**.

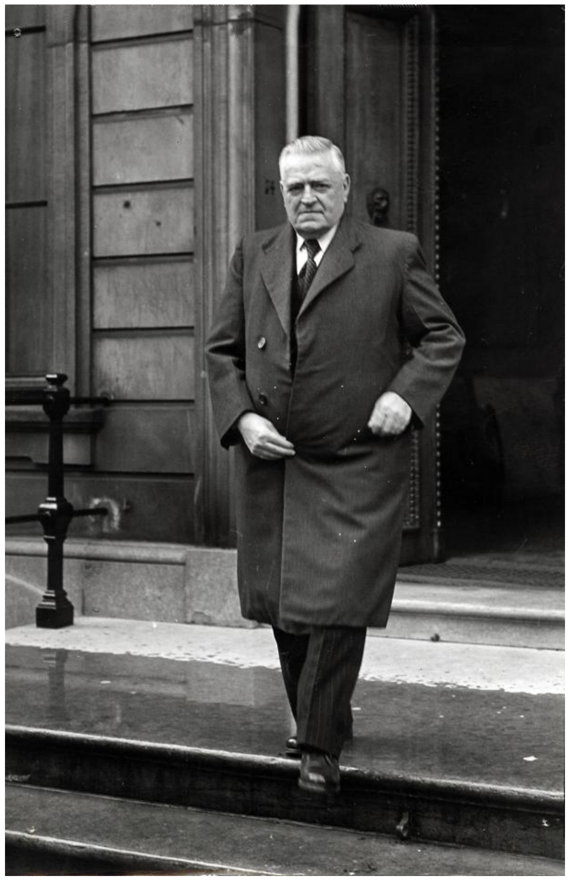
CLAIM: An integrable almost complex structure **is always formally integrable.**

Proof: Locally, the bundle $T^{1,0}(M)$ is generated by d/dz_i, where z_i are complex coordinates. These vector fields commute, hence satisfy $[d/dz_i, d/dz_j] \in T^{1,0}(M)$. This means that the Frobenius form vanishes. ■

THEOREM: (Newlander-Nirenberg) A complex structure I on M is integrable if and only if it is formally integrable.

Proof: (real analytic case) later today.

Jan Schouten (1883-1971), Kentaro Yano (1912-1993), Albert Nijenhuis (1926-2015)



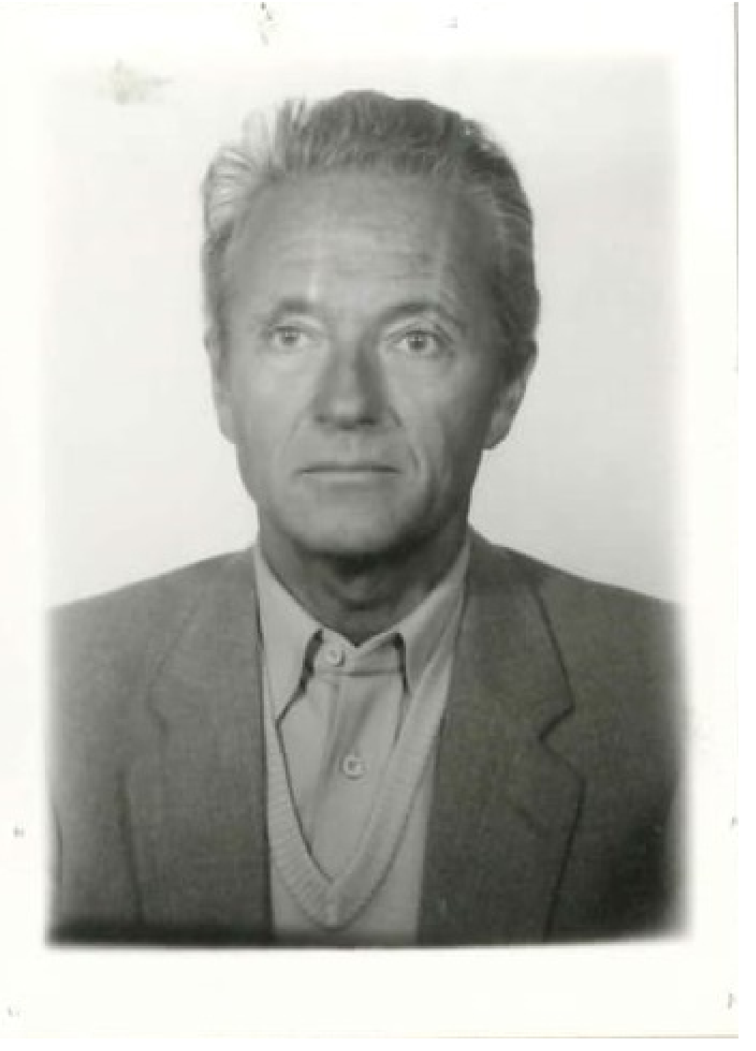
"Nijenhuis tensor" originates with the paper

J. A. Schouten and K. Yano, "On the geometrical meaning of the vanishing of the Nijenhuis tensor in an X_{2n} with an almost complex structure," Koninklijke Nederlandse Akademie van Wetenschappen. Indagationes Mathematicae **17** (1955), 133-138.

History of Newlander-Nirenberg

The real analytic version of N.-N. was obtained in 1951 by Beno Eckmann (1917-2008) and his student Alfred Frölicher (this is their only joint publication).

Beno Eckmann, 1988



Alfred Frölicher, 1983

Nota bene: everybody misspells him "Fröhlicher". This is wrong, I checked.

History of Newlander-Nirenberg (2)

Allyn Jackson, *Interview with Louis Nirenberg*, Int. Math. Nachrichten, 229 (2015), 27-41.

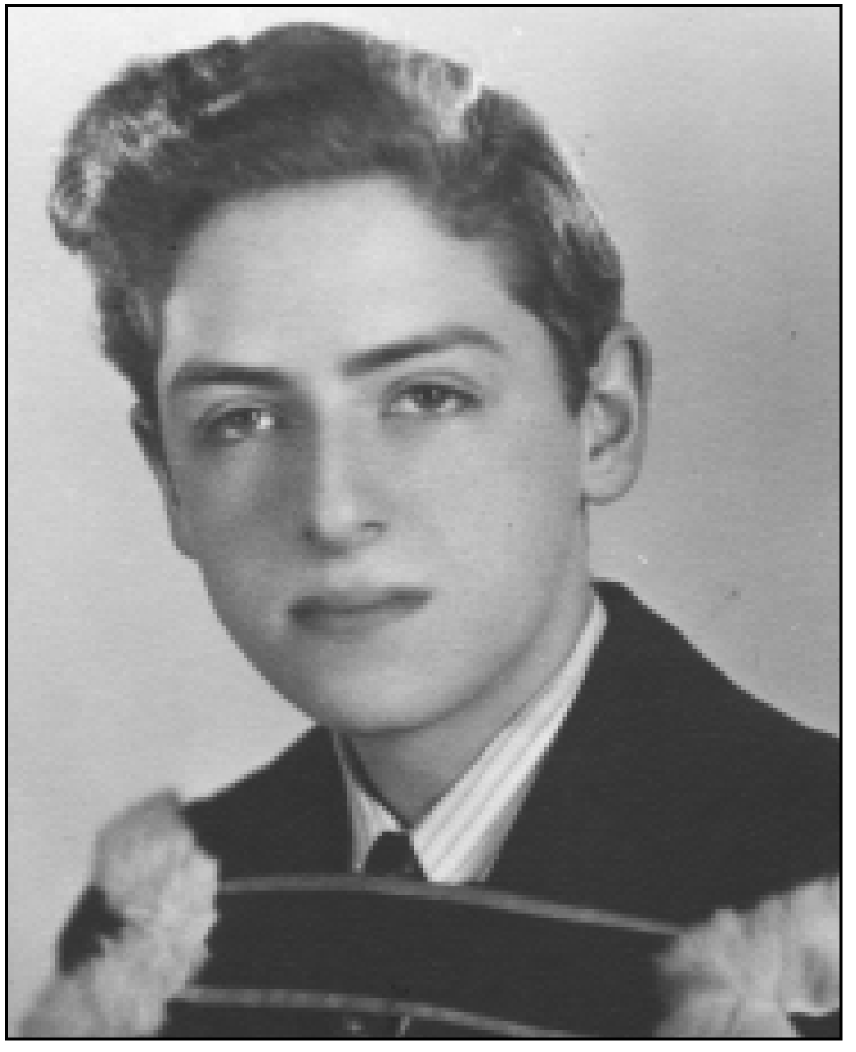
Allyn Jackson: Around 1957 you worked on the integrability problem with your student Newlander. How did that paper come about?

Louis Nirenberg: That happened in an interesting way. I heard of the problem from two people, first from André Weil. He said, "Ah, you people in partial differential equations! You're not working on the important problems! Here is an important problem that we need in complex analysis. Why aren't you working on that?" And later Chern brought the problem to my attention. So I thought, okay, let's have a stab at that. So I suggested to Newlander that we work on it. The problem, a local one, is this: In $\mathbb{R}^{2n}$, can one recognize the Cauchy-Riemann operators if they are given in some arbitrary coordinate system? For $n = 1$, the long-known answer is yes. For $n > 1$, there are necessary integrability conditions, and these turn out to be sufficient. In the problem you have, sort of, Cauchy- Riemann equations with some extra terms. The idea was to get rid of those terms one at a time. In the simplest case we first looked at, there was just one extra term, and Newlander had the idea of how to get rid of that term. Then we worked together on the more general case, first in two complex dimensions. I was sure that everything would work in higher dimensions, but then when it came to writing it down, we discovered that the idea didn't work in higher dimensions. We then came up with a different proof, which got rid of everything at the same time, but by making a fully nonlinear transformation of the whole problem. But I was drawn to the problem because of André Weil and Chern.

Allyn Jackson: Where is Newlander now?

Louis Nirenberg: He got a job in Seattle, but he had some psychological problems and he couldn't teach. After a few months he gave up his job and gave up mathematics. He moved to his hometown, Denver, Colorado, and we were in touch every New Year's for a number of years, but now I've lost touch with him. I don't know where he is. He stopped mathematics shortly after the Ph.D. thesis. That was the only paper he ever wrote.

Louis Nirenberg (1925-2020)



L. Nirenberg at the time of receiving his B.Sc. from McGill University, 1945.

Newlander, A.; Nirenberg, L. *Complex analytic coordinates in almost complex manifolds. Ann. of Math. (2) 65 (1957), 391-404.*

Frobenius theorem (reminder)

Frobenius Theorem: Let $B \subset TM$ be a sub-bundle. Then B is involutive if and only if each point $x \in M$ has a neighbourhood $U \ni x$ and **a smooth submersion $U \xrightarrow{\pi} V$ such that B is its vertical tangent space: $B = T_\pi M$.**

REMARK: The implication **"$B = T_\pi M$"** $\Rightarrow$ **"Frobenius form vanishes"** is clear because of local coordinate form of the submersions.

DEFINITION: The fibers of π are called **leaves**, or **integral submanifolds** of the distribution B. Globally on M, **a leaf of B** is a maximal connected manifold $Z \hookrightarrow M$ which is immersed to M and tangent to B at each point. A distribution for which Frobenius theorem holds is called **integrable**. If B is integrable, the set of its leaves is called **a foliation**. The leaves are manifolds which are immersed to M, but not necessarily closed.

REMARK: To prove the Frobenius theorem for $B \subset TM$, **it suffices to show that each point is contained in an integral submanifold.** In this case, the smooth submersion $U \xrightarrow{\pi} V$ is the projection to the leaf space of B.

(1,0) and (0,1)-foliations

DEFINITION: Let $B \subset TM$ be a sub-bundle. The **foliation associated with** B is a family of submanifolds $X_t \subset U$, defined for each sufficiently small subset of M, called **the leaves of the foliation**, such that B is the bundle of vectors tangent to X_t. In this case, X_t are called **the leaves** of the foliation.

REMARK: Frobenius theorem says that B **is involutive if and only if it is tangent to a foliation.**

REMARK: Let (M, I) be a real analytic almost complex manifold, and $M_{\mathbb C}$ its complexification. Replacing $M_{\mathbb C}$ by a smaller neighbourhood of M, we may assume that the tensor I is extended to an endomorphism $I: TM_{\mathbb C} \longrightarrow TM_{\mathbb C}$, $I^2 = -\operatorname{Id}$. **Since $TM_{\mathbb C}$ is a complex vector bundle, I acts there with the eigenvalues $\sqrt{-1}$ and $-\sqrt{-1}$, giving a decomposition $TM_{\mathbb C} = T^{1,0}M_{\mathbb C} \oplus T^{0,1}M_{\mathbb C}$**

DEFINITION: (1,0)-foliation is the foliation on $M_{\mathbb C}$ tangent to $T^{1,0}M_{\mathbb C}$, **(0,1)-foliation** is the foliation tangent to $T^{0,1}M_{\mathbb C}$.

(1,0)-foliation on $M_{\mathbb C} = M \times \overline{M}$.

REMARK: Let (M,I) be a integrable almost complex manifold, $M_{\mathbb C} = M\times\overline{M}$ its complexification, and $\pi,\overline{\pi}$ projections of $M_{\mathbb C}$ to M and $\overline{M}$. **Then the fibers of $\overline{\pi}$ is the (1,0)=foliation, and the fibers of π is the (0,1)-foliation.**

REMARK: Let $TM_{\mathbb C} = T'\oplus T''$ be a decomposition of $TM_{\mathbb C}$ onto part tangent to fibers of $\overline{\pi}$ and tangent to fibers of π. **On $M_{\mathbb R}$ the decomposition $TM_{\mathbb C} = T'\oplus T''$ coincides with the decomposition $TM\otimes\mathbb C = T^{1,0}M\oplus T^{0,1}M$.**

COROLLARY: Let (M,I) be a integrable almost complex manifold. **Then I is a real analytic almost complex structure.**

Proof: Extend I to an operator on $M_{\mathbb C}$ acting as $\sqrt{-1}$ on T' and $-\sqrt{-1}$ on T''. This operator is complex analytic because the decomposition $TM = T'\oplus T''$ is holomorphic. ■

Corollary 1: Let (M,I) be a real analytic almost complex manifold. Then the holomorphic functions on $M_{\mathbb C}$ which are constant on the leaves of (0,1)-foliation **restrict to holomorphic functions on $(M,I)\subset M_{\mathbb C}$.**

Proof: Such functions are constant in the $(0,1)$-direction on $TM\otimes\mathbb C$. ■

Integrability of real analytic almost complex structure

THEOREM: (Newlander-Nirenberg for real analytic manifolds) Let (M, I) be a real analytic almost complex manifold. **Then M is integrable.**

Proof. Step 1: Consider the complexification $M_{\mathbb{C}}$ of M, and let $TM_{\mathbb{C}} = T^{1,0}M_{\mathbb{C}} \oplus T^{0,1}M_{\mathbb{C}}$ be the decomposition defined above. By Frobenius theorem, there exists a foliation tangent to $T^{0,1}M_{\mathbb{C}}$ and one tangent to $T^{1,0}M_{\mathbb{C}}$. Since the leaves of these foliations are transversal, **locally $M_{\mathbb{C}}$ is a product of M' and M'' which are identified with the space of leaves of $T^{0,1}M_{\mathbb{C}}$ and $T^{1,0}M_{\mathbb{C}}$.**

Step 2: Locally, functions on M' can be lifted to $M' \times M'' = M_{\mathbb{C}}$, giving functions which are constant on the leaves of the foliation tangent to $T^{0,1}M_{\mathbb{C}}$. By Corollary 1, such functions are holomorphic on (M, I). Choose a collection of $n = \frac{1}{2}\dim_{\mathbb{R}} M$ holomorphic functions $f_1, ...f_n$ on $M_{\mathbb{C}}$ which are constant on the leaves of $T^{0,1}M_{\mathbb{C}}$ and have linearly independent differentials in $x \in M \subset M_{\mathbb{C}}$. By inverse function theorem, **$f_1, ..., f_n$ is a holomorphic coordinate system in a neigbourhood of $x \in (M, I)$,** and the transition functions between such coordinate systems are by construction holomorphic. ■