Hodge theory

lecture 4: Kähler manifolds

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Homogeneous spaces

DEFINITION: A Lie group is a smooth manifold equipped with a group structure such that the group operations are smooth. Lie group G **acts on a manifold** M if the group action is given by the smooth map $G \times M \longrightarrow M$.

DEFINITION: Let G be a Lie group acting on a manifold M transitively. Then M is called **a homogeneous space**. For any $x \in M$ the subgroup $\mathrm{St}_x(G) = \{g \in G \mid g(x) = x\}$ is called **stabilizer of a point** x, or **isotropy subgroup**.

CLAIM: For any homogeneous manifold M with transitive action of G, **one has $M = G/H$,** where $H = \mathrm{St}_x(G)$ is an isotropy subgroup.

Proof: The natural surjective map $G \longrightarrow M$ putting g to $g(x)$ identifies M with the space of conjugacy classes G/H. ■

REMARK: Let $g(x) = y$. Then $\mathrm{St}_x(G)^g = \mathrm{St}_y(G)$: **all the isotropy groups are conjugate.**

Isotropy representation

DEFINITION: Let $M = G/H$ be a homogeneous space, $x \in M$ and $\operatorname{St}_x(G)$ the corresponding stabilizer group. The **isotropy representation** is the natural action of $\operatorname{St}_x(G)$ on T_xM.

DEFINITION: A bilinear symmetric form (or any tensor) Φ on a homogeneous manifold $M = G/H$ is called **invariant** if it is mapped to itself by all diffeomorphisms which come from $g \in G$.

REMARK: Let Φ_x be an isotropy invariant tensor on T_xM, where $M = G/H$ is a homogeneous space. For any $y \in M$ obtained as $y = g(x)$, consider the form Φ_y on T_yM obtained as $\Phi_y := g^*(\Phi)$. The choice of g is not unique, however, for another $g' \in G$ which satisfies $g'(x) = y$, we have $g = g'h$ where $h \in \operatorname{St}_x(G)$. Since Φ is h-invariant, **the tensor Φ_y is independent from the choice of g.**

We proved

THEOREM: Let $M = G/H$ be a homogeneous space and $x \in M$ a point. Then G-invariant tensors on $M = G/H$ **are in bijective correspondence with isotropy invariant tensors** on the vector space T_xM. $\blacksquare$

Kähler manifolds

DEFINITION: An Riemannian metric g on an almost complex manifiold M is called **Hermitian** if $g(Ix,Iy) = g(x,y)$. In this case, $g(x,Iy) = g(Ix,I^2y) = -g(y,Ix)$, hence $\omega(x,y) := g(x,Iy)$ is skew-symmetric.

REMARK: Given any Riemannian metric g on an almost complex manifold, **a Hermitian metric h can be obtained as $h = g + I(g)$, where $I(g)(x,y) = g(I(x),I(y))$.**

DEFINITION: The differential form $\omega \in \Lambda^{1,1}(M)$ is called **the Hermitian form** of (M,I,g).

REMARK: It is $U(1)$-invariant, hence **of Hodge type (1,1)**.

REMARK: In the triple I, g, ω, **each element can recovered from the other two.**

DEFINITION: A complex Hermitian manifold (M,I,ω) is called **Kähler** if $d\omega = 0$. The cohomology class $[\omega] \in H^2(M)$ of a form ω is called **the Kähler class** of M, and ω **the Kähler form**.

Erich Kähler

(Erich Kähler: 1990)

16 January 1906 - 31 May 2000

Chez les Weil. André et Simone

André Weil: 6 May 1906 - 6 August 1998.

"Simone et André à Penthiévre, 1918-1919"

Representations acting transitively on a sphere

THEOREM: Let G be a group acting on a vector space V. Suppose that G acts transitively on a unit sphere $\{x \in V \mid g(x) = 1\}$. **Then a G-invariant bilinear symmetric form is unique up to a constant multiplier.**

Proof. Step 1: Since G preserves the sphere, which is a level set of the quadratic form g, g is G-invariant.

Step 2: For any G-invariant quadratic form g', the function $x \longrightarrow \frac{g'(x)}{g(x)}$ is constant on spheres and invariant under homothety, hence it is constant. ∎

EXERCISE: Let V be a representation of G, and suppose G acts transitively on a sphere. **Prove that V is an irreducible representation.**

EXERCISE: Prove the **Schur lemma:** let V be an irreducible representation of G over $\mathbb{R}$, and g a G-invariant positive definite bilinear symmetric form. **Then any G-invariant bilinear symmetric form is proportional to g.**

Fubini-Study form

EXAMPLE: Consider the natural action of the unitary group $U(n+1)$ on $\mathbb{C}P^n$. The stabilizer of a point is $U(n)\times U(1)$.

THEOREM: There exists an $U(n+1)$-invariant Riemannian form on $\mathbb{C}P^n$. Moreover, **such a form is unique up to a constant multiplier, and Kähler.**

REMARK: This Riemannian structure is called **the Fubini-Study metric**, and its Hermitian form **the Fubini-Study form**.

Proof. Step 1: To construct a $U(n+1)$-invariant Riemannian form on $\mathbb{C}P^n$, we take a $U(n)$-invariant form on $T_x\mathbb{C}P^n$ and apply Theorem 1. A $U(n)$-invariant form on $T_x\mathbb{C}P^n$ exists, because it is a standard representation.

Step 2: Uniqueness follows because the isotropy group acts transitively on a sphere. ■

CLAIM: **The Fubini-Study form is closed,** and the corresponding metric is Kähler.

Proof: Let ω be a Fubini-Study form. Then $d\omega$ is an isotropy-invariant 3-form on $T_x\mathbb{C}P^n$. However, the isotropy group contains $-\operatorname{Id}$, **hence all isotropy-invariant odd tensors vanish.** ■

Projective manifolds

DEFINITION: Let M be a complex manifold, and $X \subset M$ a smooth submanifold. It is called **a complex submanifold** if $I(TX) \subset TX$, and the map $X \hookrightarrow M$ **a complex embedding**. A complex manifold which admits a complex embedding to $\mathbb{C}P^n$ is called **a projective manifold**.

REMARK: A complex submanifold of a Kähler manifold is Kähler. Indeed, restriction of a Hermitian metric is Hermitian, and restriction of a closed form is closed. Therefore, **all projective manifolds are Kähler**.

DEFINITION: A subvariety of $\mathbb{C}P^n$ is called **complex algebraic** if can be obtained as common zeroes of a system of homogeneous polynomial equations.

THEOREM: (Chow theorem) All complex submanifolds in $\mathbb{C}P^n$ are complex algebraic.

Kodaira embedding theorem

DEFINITION: **Kähler class** of a Kähler manifold is the cohomology class $[\omega] \in H^2(M,\mathbb{R})$ of its Kähler form. We say that M **has integer Kähler class** if $[\omega]$ belongs to the image of $H^2(M,\mathbb{Z})$ in $H^2(M,\mathbb{R})$

REMARK: $H^2(\mathbb{C}P^n,\mathbb{R}) = \mathbb{R}$. This implies that **the cohomology class of Fubini-Study form can be chosen integer.** In particular, **all projective manifolds admit Kähler structures with integer Kähler classes.**

THEOREM: **(Kodaira embedding theorem)** Let M be a compact Kähler manifold with an integer Kähler class. **Then it is projective.**

This theorem will be proven later in these lectures.

Classes of almost complex manifolds