

Hodge theory

lecture 5: Levi-Civita connection

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Connections

DEFINITION: Recall that **a connection** on a bundle B is an operator $\nabla : B \longrightarrow B \otimes \Lambda^1 M$ satisfying $\nabla(fb) = b \otimes df + f\nabla(b)$, where $f \longrightarrow df$ is de Rham differential. When X is a vector field, we denote by $\nabla_X(b) \in B$ the term $\langle \nabla(b), X \rangle$.

REMARK: A connection ∇ on B gives a connection $B^* \xrightarrow{\nabla^*} \Lambda^1 M \otimes B^*$ on the dual bundle, by the formula

$$d(\langle b, \beta \rangle) = \langle \nabla b, \beta \rangle + \langle b, \nabla^* \beta \rangle$$

These connections are usually denoted **by the same letter ∇** .

REMARK: For any tensor bundle $\mathcal{B}_1 := B^* \otimes B^* \otimes \dots \otimes B^* \otimes B \otimes B \otimes \dots \otimes B$ **a connection on B defines a connection on $\mathcal{B}_1$** using the Leibniz formula:

$$\nabla(b_1 \otimes b_2) = \nabla(b_1) \otimes b_2 + b_1 \otimes \nabla(b_2).$$

Parallel transport along the connection

THEOREM: Let B be a vector bundle with connection over $\mathbb{R}$. Then for each $x \in \mathbb{R}$ and each vector $b_x \in B|_x$ **there exists a unique section $b \in B$ such that $\nabla b = 0$, $b|_x = b_x$.**

Proof: This is existence and uniqueness of solutions of an ODE $\frac{db}{dt} + A(b) = 0$.

■

DEFINITION: Let $\gamma : [0, 1] \rightarrow M$ be a smooth path in M connecting x and y , and (B, ∇) a vector bundle with connection. Restricting (B, ∇) to $\gamma([0, 1])$, we obtain a bundle with connection on an interval. Solve an equation $\nabla(b) = 0$ for $b \in B|_{\gamma([0, 1])}$ and initial condition $b|_x = b_x$. This process is called **parallel transport** along the path via the connection. The vector $b_y := b|_y$ is called **vector obtained by parallel transport of b_x along γ** . **Holonomy group** of γ is the group of endomorphisms of the fiber B_x obtained from parallel transports along all paths starting and ending in $x \in M$.

Lie algebra and tensors

DEFINITION: Let V be a representation of a Lie algebra $\mathfrak{g}$. **Then V^* is also a representation;** the action of $\mathfrak{g}$ on V^* is given by the formula $\langle g(x), \lambda \rangle = -\langle x, g(\lambda) \rangle$, for all $x \in V, \lambda \in V^*$. A tensor product of two $\mathfrak{g}$ -representations V_1, V_2 is also a $\mathfrak{g}$ -representation, with the action of $\mathfrak{g}$ defined by $g(x \otimes y) = g(x) \otimes y + x \otimes g(y)$. This defines the action of $\mathfrak{g}$ on all tensor powers $V^{\otimes i} \otimes (V^*)^{\otimes j}$, which are called **the tensor representations** of $\mathfrak{g}$. We say that $\mathfrak{g}$ **preserves a tensor Φ** if $g(\Phi) = 0$ for all $g \in \mathfrak{g}$.

EXAMPLE: The algebra of all $g \in \text{End}(V)$ preserving a non-degenerate bilinear symmetric form $h \in \text{Sym}^2(V^*)$ is called **orthogonal algebra**, denoted $\mathfrak{so}(V, h)$ or $\mathfrak{so}(V)$. Since $g \in \mathfrak{so}(V)$ if and only if $h(g(x), y) = -h(x, g(y))$, **$\mathfrak{so}(V)$ is represented by antisymmetric matrices.**

CLAIM: Let $h \in \text{Sym}^2(V^*)$ be a non-degenerate bilinear symmetric form. Using h , we identify V and V^* . This gives an isomorphism $V^* \otimes V^* \xrightarrow{\tau} V^* \otimes V = \text{End}(V)$. **Then $\tau(\wedge^2 V^*) = \mathfrak{so}(V)$.**

Proof: For any $f \in \text{End}(V)$, the 2-form $\tau^{-1}(f)$ is written as $x, y \longrightarrow h(f(x), y)$. By definition, $f \in \mathfrak{so}(V)$ means that $h(f(x), y) = -h(x, f(y))$ and this happens if and only if $\tau^{-1}(f)$ is antisymmetric. ■

Parallel tensors

DEFINITION: Let B be a vector bundle, and $\psi \in B^{\otimes i} \otimes (B^*)^{\otimes j}$ a tensor on B . We say that **connection ∇ preserves ψ** if $\nabla(\psi) = 0$. In this case we also say that ψ is **parallel** with respect to the connection.

REMARK: $\nabla(\psi) = 0$ is equivalent to ψ being a solution of $\nabla(\psi) = 0$ on each path γ . This means that **parallel transport preserves ψ** .

We obtained

COROLLARY: **A tensor is parallel if and only if it is holonomy invariant.**

EXAMPLE: Orthogonal connection: given a positive definite form $h \in \text{Sym}^2 B^*$ on B , a connection ∇ such that $\nabla(h) = 0$ is called **orthogonal**.

EXAMPLE: Suppose that (B, I) is a complex vector bundle equipped with a Hermitian metric h . A connection ∇ such that $\nabla(I) = \nabla(h) = 0$ is called **unitary**, or **Hermitian**.

Torsors and affine spaces

DEFINITION: A **torsor** over a group G is a space X equipped with a free and transitive action of G , $g, x \longrightarrow \rho(g, x)$.

DEFINITION: **Morphism** of torsors $(X, G, \rho) \xrightarrow{\Psi} (X', G', \rho')$ is a pair $\Psi_X : X \longrightarrow X', \Psi_G : G \longrightarrow G'$, where Ψ_G is a group homomorphism satisfying $\Psi_X(\rho(g, x)) = \rho'(\Psi_G(g), \Psi_X(x))$ (that is, compatible with the map Ψ_X).

REMARK: This defines the category of torsors.

DEFINITION: **Affine space** is a torsor over a vector space V , which is called **linearization**. The action of V on A is denoted $a, v \longrightarrow a \dot{+} v$.

EXAMPLE: Given two connections ∇ and ∇_1 on B , the difference $\nabla - \nabla_1$ is an $\text{End}(B)$ -valued 1-form. Converse is also true: for any $\text{End}(B)$ -valued 1-form $A \in \Lambda^1 M \otimes \text{End}(B)$, the operator $\nabla \dot{+} A$ is a connection. In other words, **the space of connections is an affine space over $\Lambda^1 M \otimes \text{End}(B)$.**

Affine space of orthogonal connections

CLAIM: Let B be a bundle with a scalar product. Then **the space of orthogonal connections on B is an affine space over $\Lambda^1 M \otimes \mathfrak{so}(B)$.**

Proof: Let $s \in B^* \otimes B^*$ be a 2-form on B . The action of $A := \nabla - \nabla_1$ on $B^* \otimes B^*$ is given by $A(s)(x, y) = -s(A(x), y) - s(x, A(y))$. Therefore, a difference A of orthogonal connections satisfies $h(A(x), y) = -h(x, A(y))$ for all $x, y \in B$. This is the same as $A \in \Lambda^1 M \otimes \mathfrak{so}(B)$. ■

Similarly one proves

CLAIM: Let B be a bundle with a Hermitian structure and a tensor Φ , and $\mathfrak{g} \subset \text{End}(B)$ the Lie algebra of endomorphisms preserving Φ . Then **the space of connections on B preserving Φ is an affine space over $\Lambda^1 M \otimes \mathfrak{g}$.**

REMINDER: de Rham algebra

DEFINITION: Let Λ^*M denote the vector bundle with the fiber $\Lambda^*T_x^*M$ at $x \in M$ ($\Lambda^*T_x^*M$ is the Grassman algebra of the cotangent space T_x^*M). The sections of $\Lambda^i M$ are called **differential i -forms**. The algebraic operation “wedge product” defined on differential forms is $C^\infty M$ -linear; the space Λ^*M of all differential forms is called **the de Rham algebra**.

REMARK: $\Lambda^0 M = C^\infty M$.

THEOREM: There exists a unique operator $C^\infty M \xrightarrow{d} \Lambda^1 M \xrightarrow{d} \Lambda^2 M \xrightarrow{d} \Lambda^3 M \xrightarrow{d} \dots$ satisfying the following properties

1. On functions, d is equal to the differential.
2. $d^2 = 0$
3. $d(\eta \wedge \xi) = d(\eta) \wedge \xi + (-1)^{\tilde{\eta}} \eta \wedge d(\xi)$, where $\tilde{\eta} = 0$ where $\eta \in \Lambda^{2i} M$ is **an even form**, and $\eta \in \Lambda^{2i+1} M$ is **odd**.

DEFINITION: The operator d is called **de Rham differential**.

Cartan formula

CLAIM: For any $\eta \in \Lambda^1 M$, and $X, Y \in TM$ one has

$$d\eta(X, Y) = \eta([X, Y]) - \text{Lie}_X(\eta(Y)) + \text{Lie}_Y(\eta(X)).$$

Proof: Two sides of this equation define two operators $d, d_1 : \Lambda^1 M \rightarrow \Lambda^2 M$. Both operators satisfy the Leibniz rule $d(f\eta) = df \wedge \eta + f d\eta$. When $\eta = df$ is exact, one has

$$\begin{aligned} \eta([X, Y]) - \text{Lie}_X(\eta(Y)) + \text{Lie}_Y(\eta(X)) &= \\ &= \text{Lie}_{[X, Y]}(f) - \text{Lie}_X \text{Lie}_Y(f) + \text{Lie}_Y \text{Lie}_X(f) = 0 \end{aligned}$$

hence $d_1(\alpha) = 0$ on all closed forms. A map $\delta : \Lambda^1(M) \rightarrow \Lambda^2(M)$ which vanishes on closed forms and satisfies the Leibniz rule is de Rham differential, which can be seen from the axiomatic definition of d . ■

Torsion

DEFINITION: Let ∇ be a connection on $\Lambda^1 M$,

$$\Lambda^1 \xrightarrow{\nabla} \Lambda^1 M \otimes \Lambda^1 M.$$

Torsion of ∇ $T_\nabla : \Lambda^1 M \longrightarrow \Lambda^2 M$ is a map $\nabla \circ \text{Alt} - d$, where $\text{Alt} : \Lambda^1 M \otimes \Lambda^1 M \longrightarrow \Lambda^2 M$ is exterior multiplication.

REMARK:

$$\begin{aligned} T_\nabla(f\eta) &= \text{Alt}(f\nabla\eta + df \otimes \eta) - d(f\eta) \\ &= f \left[\text{Alt}(\nabla\eta) - d\eta \right] + df \wedge \eta - df \wedge \eta = fT_\nabla(\eta). \end{aligned}$$

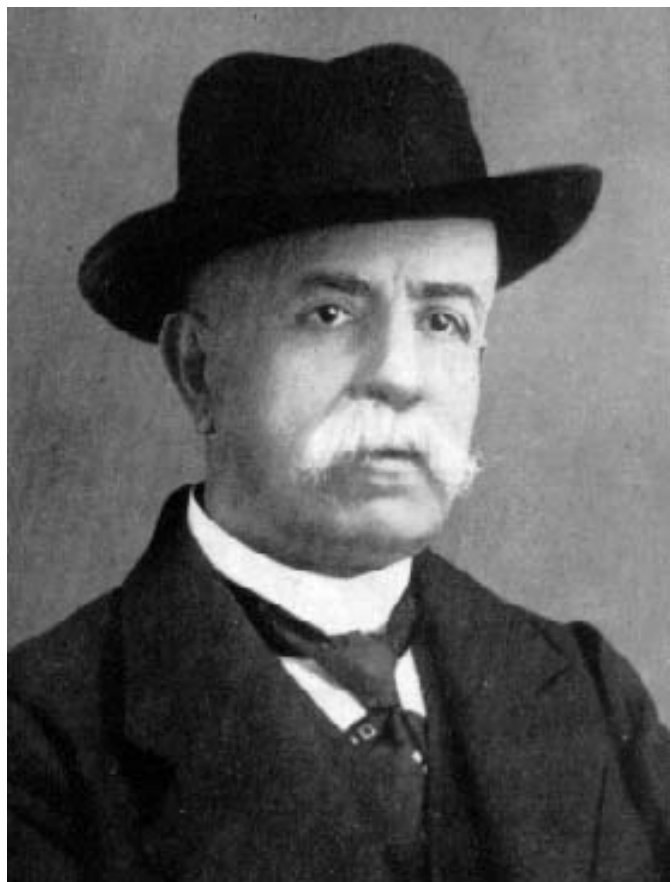
Therefore T_∇ is linear.

DEFINITION: Let (M, g) be a Riemannian manifold. A connection ∇ on TM is called **orthogonal** if $\nabla(g) = 0$, and **Levi-Civita connection** if it is orthogonal and has zero torsion.

THEOREM: (“the fundamental theorem of Riemannian geometry”) Every Riemannian manifold admits a Levi-Civita connection, and it is unique.

Will be proven later today.

Gregorio Ricci-Curbastro, Tullio Levi-Civita



Gregorio Ricci-Curbastro,
1853-1925



Tullio Levi-Civita,
1873-1941

...With his former student Tullio Levi-Civita, he wrote his most famous single publication, a pioneering work on the calculus of tensors, signing it as Gregorio Ricci. This appears to be the only time that Ricci-Curbastro used the shortened form of his name in a publication, and continues to cause confusion.

Torsion and commutator of vector fields

REMARK: Cartan formula gives

$$\begin{aligned} T_{\nabla}(\eta)(X, Y) &= \nabla_X(\eta)(Y) - \nabla_Y(\eta)(X) - d\eta(X, Y) \\ &= \nabla_X(\eta)(Y) - \nabla_Y(\eta)(X) - \eta([X, Y]) - \text{Lie}_X(\eta(Y)) + \text{Lie}_Y(\eta(X)). \end{aligned}$$

On the other hand, $\nabla_X(\eta)(Y) = \text{Lie}_X(\eta(Y)) - \eta(\nabla_X(Y))$. Comparing the equations, we obtain

$$T_{\nabla}(\eta)(X, Y) = \eta\left(\nabla_X(Y) - \nabla_Y(X) - [X, Y]\right).$$

Torsion is often defined as a map $\wedge^2 TM \longrightarrow TM$ using the formula $\nabla_X(Y) - \nabla_Y(X) - [X, Y]$.

We have just proved

CLAIM: The torsion tensor $\nabla_X(Y) - \nabla_Y(X) - [X, Y]$ is dual to the torsion $\nabla \circ \text{Alt} - d : \wedge^1 M \longrightarrow \wedge^2 M$ defined above. ■

Linearization of the torsion

REMARK: Consider the space $\mathcal{A}(\Lambda^1 M)$ of connections on $\Lambda^1 M$. The torsion defines an affine map

$$\mathcal{A}(\Lambda^1 M) \longrightarrow \text{Hom}(\Lambda^1 M, \Lambda^2 M) = TM \otimes \Lambda^2 M.$$

because $T(\nabla + \alpha) = T(\nabla) + \text{Alt}_{12}(\alpha)$, where $\text{Alt}_{12} : \Lambda^1 M \otimes \text{End}(\Lambda^1 M) \longrightarrow \Lambda^2 M \otimes TM$ is antisymmetrization in the first two indices.

DEFINITION: **Linearized torsion** is a map

$$T_{lin} : \Lambda^1(M) \otimes \Lambda^1(M) \otimes TM \longrightarrow \Lambda^2 M \otimes TM$$

obtained as a linearization of the torsion map. **It is equal to Alt_{12} .**

Existence of orthogonal connections

CLAIM: Let B be a vector bundle equipped with a scalar product. **Then B admits an orthogonal connection.**

Proof: Chose a covering $\{U_i\}$, such that B is trivial on each U_i and admits an orthonormal basis in each U_i . On each U_i we chose a connection ∇_i preserving this basis. Let ψ_i be a partition of unit subjugated to $\{U_i\}$. Then **the formula $\nabla(b) := \sum \nabla_i(\psi_i b)$ defines an orthogonal connection.**

THEOREM: (“the fundamental theorem of Riemannian geometry”) **Every Riemannian manifold admits a Levi-Civita connection, and it is unique.**

Proof: See the next slide.

Levi-Civita connection (existence and uniqueness)

Proof. Step 1: Chose an orthogonal connection ∇_0 on $\Lambda^1 M$. The space $\mathcal{A}$ of orthogonal connections is affine and **its linearization is $\Lambda^1 M \otimes \mathfrak{so}(TM)$** . We shall identify $\mathfrak{so}(TM)$ and $\Lambda^2 M$. Then **$\mathcal{A}$ is an affine space over $\Lambda^1 M \otimes \Lambda^2 M$** .

Step 2: Then the linearized torsion map is

$$T_{lin} : \Lambda^1 M \otimes \mathfrak{so}(TM) = \Lambda^1(M) \otimes \Lambda^2 M \xrightarrow{\text{Alt}_{12}} \Lambda^2 M \otimes \Lambda^1 M = \Lambda^2 M \otimes TM.$$

It is an isomorphism. Indeed, on the right and on the left there are bundles of the same rank, hence it would suffice to show that $T_{lin} = \text{Alt}_{12}$ is injective. However, if $\eta \in \ker T_{lin}$, it is a form which is symmetric on first two arguments and antisymmetric on the second two, giving $\eta(x, y, z) = \eta(y, x, z) = -\eta(y, z, x)$. This gives $\sigma(\eta) = -\eta$, where σ is a cyclic permutation of the arguments. Since $\sigma^3 = 1$, this implies $\eta = 0$.

Step 3: We have shown that **an orthogonal connection is uniquely determined by its torsion**. Indeed, torsion map is an isomorphism of affine spaces.

Step 4: Let $\nabla := \nabla_0 - T_{lin}^{-1}(T_{\nabla_0})$. Then $T_{\nabla} = T_{\nabla_0} - T_{lin}(T_{lin}^{-1}(T_{\nabla_0})) = 0$, hence **∇ is torsion-free.** ■