Metric spaces 1: Remedial topology

Rules: It is recommended to solve all unmarked and !-problems or to find the solution online. It's better to do it in order starting from the beginning, because the solutions are often contained in previous problems. The problems with * are harder, and ** are very hard; don't be disappointed if you can't solve them, but feel free to try. Have fun!

1.1 Topological spaces

Definition 1.1. A set of all subsets of M is denoted 2^M. **Topology** on M is a collection of subsets $S \subset 2^M$ called **open subsets**, and satisfying the following conditions.

1. Empty set and M are open
2. A union of any number of open sets is open
3. An intersection of a finite number of open subsets is open.

A complement of an open set is called **closed**. A set with topology on it is called **a topological space**. **An open neighbourhood** of a point is an open set containing this point.

Definition 1.2. A map $\phi: M \longrightarrow M'$ of topological spaces is called **continuous** if a preimage of each open set $U \subset M'$ is open in M. A bijective continuous map is called **a homeomorphism** if its inverse is also continuos.

Exercise 1.1. Let M be a set, and S a set of all subsets of M. Prove that S defines topology on M. This topology is called **discrete**. Describe the set of all continuous maps from M to a given topological space.

Exercise 1.2. Let M be a set, and $S \subset 2^M$ a set of two subsets: empty set and M. Prove that S defines topology on M. This topology is called **codiscrete**. Describe the set of all continuous maps from M to a space with discrete topology.

Definition 1.3. Let M be a topological space, and $Z \subset M$ its subset. **Open subsets** of Z are subsets obtained as $Z \cap U$, where U is open in M. This topology is called **induced topology**.

Definition 1.4. A metric space is a set M equipped with a **distance function** $d: M \times M \longrightarrow \mathbb{R}^{\geqslant 0}$ satisfying the following axioms.

1. $d(x, y) = 0$ iff $x = y$.
2. $d(x, y) = d(y, x)$.
3. (triangle inequality) $d(x, y) + d(y, z) \geqslant d(x, z)$.

An **open ball** of radius r with center in x is $\{y \in M \mid d(x, y) < r\}$.

Definition 1.5. Let M be a metric space. A subset $U \subset M$ is called **open** if it is obtained as a union of open balls. This topology is called **induced by the metric**.

Definition 1.6. A topological space is called **metrizable** if its topology can be induced by a metric.

Exercise 1.3. Show that discrete topology can be induced by a metric, and codiscrete cannot.

Exercise 1.4. Prove that an intersection of any collection of closed subsets of a topological space is closed.

Definition 1.7. An intersection of all closed supersets of $Z \subset M$ is called **closure** of Z.

Definition 1.8. A limit point of a set $Z \subset M$ is a point $x \in M$ such that any neighbourhood of M contains a point of Z other than x. **A limit** of a sequence $\{x_i\}$ of points in M is a point $x \in M$ such that any neighbourhood of $x \in M$ contains all x_i for all i except a finite number. A sequence which has a limit is called **convergent**.

Exercise 1.5. Show that a closure of a set $Z \subset M$ is a union of Z and all its limit points.

Exercise 1.6. Let $f : M \longrightarrow M'$ be a continuous map of topological spaces. Prove that $f(\lim_i x_i) = \lim_i f(x_i)$ for any convergent sequence $\{x_i \in M\}$.

Exercise 1.7. Let $f : M \longrightarrow M'$ be a map of metrizable topological spaces, such that $f(\lim_i x_i) = \lim_i f(x_i)$ for any convergent sequence $\{x_i \in M\}$. Prove that f is continuous.

Exercise 1.8 (*). Find a counterexample to the previous problem for non-metrizable, Hausdorff topological spaces (see the next subsection for a definition of Hausdorff).

Exercise 1.9 (**). Let $f: M \longrightarrow M'$ be a map of countable topological spaces, such that $f(\lim_i x_i) = \lim_i f(x_i)$ for any convergent sequence $\{x_i \in M\}$. Prove that f is continuous, or find a counterexample.

Exercise 1.10 (*). Let $f: M \longrightarrow N$ be a bijective map inducing homeomorphisms on all countable subsets of M. Show that it is a homeomorphism, or find a counterexample.

1.2 Hausdorff spaces

Definition 1.9. Let M be a topological space. It is called **Hausdorff**, or **separable**, if any two distinct points $x \neq y \in M$ can be **separated** by open subsets, that is, there exist open neighbourhoods $U \ni x$ and $V \in y$ such that $U \cap V = \emptyset$.

Remark 1.1. In topology, the Hausdorff axiom is usually assumed by default. In subsequent handouts, it will be always assumed (unless stated otherwise).

Exercise 1.11. Prove that any subspace of a Hausdorff space with induced topology is Hausdorff.

Exercise 1.12. Let M be a Hausdorff topological space. Prove that all points in M are closed subsets.

Exercise 1.13. Let M be a topological space, with all points of M closed. Prove that M is Hausdorff, or find a counterexample.

Exercise 1.14. Count the number of non-isomorphic topologies on a finite set of 4 elements. How many of these topologies are Hausdorff?

Exercise 1.15 (!). Let Z_1, Z_2 be non-intersecting closed subsets of a metrizable space M. Find open subsets $U \supset Z_1$, $V \supset Z_2$ which do not intersect.

Definition 1.10. Let M, N be topological spaces. **Product topology** is a topology on $M \times N$, with open sets obtained as unions $\bigcup_\alpha U_\alpha \times V_\alpha$, where U_α is open in M and V_α is open in N.

Exercise 1.16. Prove that a topology on X is Hausdorff if and only if the diagonal $\{(x,y) \in X \times X \ \mid \ x = y\}$ is closed in the product topology.

Definition 1.11. Let $\sim$ be an equivalence relation on a topological space M. **Factor-topology** (or **quotient topology**) is a topology on the set $M/\sim$ of equivalence classes such that a subset $U \subset M/\sim$ is open whenever its preimage in M is open.

Exercise 1.17. Let G be a finite group acting on a Hausdorff topological space M.[1] Prove that the quotient map is closed.[2]

Exercise 1.18 (*). Let $\sim$ be an equivalence relation on a topological space M, and $\Gamma \subset M \times M$ **its graph**, that is, the set $\{(x,y) \in M \times M \ \mid \ x \sim y\}$. Suppose that the map $M \longrightarrow M/\sim$ is open, and the Γ is closed in $M \times M$. Show that $M/\sim$ is Hausdorff.

Hint. Prove that diagonal is closed in $M \times M$.

Exercise 1.19 (!). Let G be a finite group acting on a Hausdorff topological space M. Prove that M/G with the quotient topology is Hausdorf,

a. (!) when M is compact

b. (*) for arbitrary M.

Hint. Use the previous exercise.

Exercise 1.20 (**). Let $M = \mathbb{R}$, and $\sim$ an equivalence relation with at most 2 elements in each equivalence class. Prove that $\mathbb{R}/\sim$ is Hausdorff, or find a counterexample.

Exercise 1.21 (*). ("gluing of closed subsets") Let M be a metrizable topological space, and $Z_i \subset M$ a finite number closed subsets which do not intersect, grouped into pairs of homeomorphic $Z_i \sim Z_i'$. Let $\sim$ an equivalence relation generated by these homeomorphisms. Show that $M/\sim$ is Hausdorff.

[1] Speaking of a group acting on a topological space, one always means continuous action.
[2] a **closed map** is a map which puts closed subsets to closed subsets.

1.3 Compact spaces

Definition 1.12. **A cover** of a topological space M is a collection of open subsets $\{U_\alpha \in 2^M\}$ such that $\bigcup U_\alpha = M$. **A subcover** of a cover $\{U_\alpha\}$ is a subset $\{U_\beta\} \subset \{U_\alpha\}$. A topological space is called **compact** if any cover of this space has a finite subcover.

Exercise 1.22. Let M be a compact topological space, and $Z \subset M$ a closed subset. Show that Z is also compact.

Exercise 1.23. Let M be a countable, metrizable topological space. Show that either M contains a converging sequence of pairwise different elements, or M is discrete.

Definition 1.13. A topological space is called **sequentially compact** if any sequence $\{z_i\}$ of points of M has a converging subsequence.

Exercise 1.24. Let M be metrizable a compact topological space. Show that M is sequentially compact.

Hint. Use the previous exercise.

Remark 1.2. **Heine-Borel theorem** says that the converse is also true: any metric space which is sequentially compact, is also compact. Its proof is moderately difficult (please check Wikipedia or any textbook on point-set topology, metric geometry or analysis; "Metric geometry" by Burago-Burago-Ivanov is probably the best place).

In subsequent handouts, you are allowed to use this theorem without a proof.

Exercise 1.25 (*). Construct an example of a Hausdorff topological space which is sequentially compact, but not compact.

Exercise 1.26 (*). Construct an example of a Hausdorff topological space which is compact, but not sequentially compact.

Definition 1.14. A **topological group** is a topological space with group operations $G\times G \longrightarrow G$, $x, y \mapsto xy$ and $G \longrightarrow G$, $x \mapsto x^{-1}$ which are continuous. In a similar way, one defines **topological vector spaces**, **topological rings** and so on.

Exercise 1.27 (*). Let G be a compact topological group, acting on a topological space M in such a way that the map $M \times G \longrightarrow M$ is continuous. Prove that the quotient space is Hausdorff.

Exercise 1.28. Let $f: X \longrightarrow Y$ be a continuous map of topological spaces, with X compact. Prove that $f(X)$ is also compact.

Exercise 1.29. Let $Z \subset Y$ be a compact subset of a Hausdorff topological space. Prove that it is closed.

Exercise 1.30. Let $f: X \longrightarrow Y$ be a continuous, bijective map of topological spaces, with X compact and Y Hausdorff. Prove that it is a homeomorphism.

Definition 1.15. A topological space M is called **pseudocompact** if any continuous function $f: M \longrightarrow \mathbb{R}$ is bounded.

Exercise 1.31. Prove that any compact topological space is pseudocompact.

Hint. Use the previous exercise.

Exercise 1.32. Show that for any continuous function $f: M \longrightarrow \mathbb{R}$ on a compact space there exists $x \in M$ such that $f(x) = \sup_{z \in M} f(z)$.

Exercise 1.33. Consider $\mathbb{R}^n$ as a metric space, with the standard (Euclidean) metric. Let $Z \subset \mathbb{R}^n$ be a closed, bounded set ("bounded" means "contained in a ball of finite radius"). Prove that Z is sequentially compact.

Exercise 1.34 (**). Find a pseudocompact Hausdorff topological space which is not compact.

Definition 1.16. A map of topological spaces is called **proper** if a preimage of any compact subset is always compact.

Exercise 1.35 (*). Let $f: X \longrightarrow Y$ be a continuous, proper, bijective map of metrizable topological spaces. Prove that f is a homeomorphism, or find a counterexample.

Exercise 1.36 (*). Let $f: X \longrightarrow Y$ be a continuous, proper map of metrizable topological spaces. Show that f is closed, or find a counterexample.