Metric spaces 2: Intrinsic metrics

Rules: You may give me the written solutions some time during this week, and I will grade them. The results might have some (very small, if any) influence on your final grade. Have fun!

Exercise 2.1. Let d be an intrinsic metric on $\mathbb{R}^n$ invariant under the action of the group of affine isometries $\mathbb{R}^n \rtimes SO(n)$. Prove that this metric is the standard Euclidean metric on $\mathbb{R}^n$.

Exercise 2.2. Let $\phi: \mathbb{R}^{\geqslant 0} \longrightarrow \mathbb{R}^{\geqslant 0}$ be a monotonous function, fixing 0 and mapping $\mathbb{R}^{>0} \longrightarrow \mathbb{R}^{>0}$. Assume that ϕ is subadditive, that is, $\phi(x+y) \leqslant \phi(x) + \phi(y)$.

a. Prove that for all metric spaces (M, d), the function $d_\phi(x,y) := \phi(d(x,y))$ defines a metric on M.

b. Let d_1 be an $\mathbb{R}^n \rtimes SO(n)$-invariant metric on $\mathbb{R}^n$. Prove that $d_1 = d_\phi$, where d_ϕ is obtained as above from the standard Euclidean metric d and some subadditive function $\phi: \mathbb{R}^{>0} \longrightarrow \mathbb{R}^{>0}$.

Exercise 2.3 (*). Let d be an intrinsic metric on a sphere S^n invariant under the action of the orthogonal group $SO(n+1)$. Prove that this metric is the standard Riemannian metric on a sphere.

Exercise 2.4. Let $\phi: [0,1] \longrightarrow \mathbb{R}^2$ be a rectifiable path. Prove that its image is a measure 0 set.

Exercise 2.5 (*). Let M be a metric space which contains a non-constant path. Prove that M contains a non-rectifiable path.

Exercise 2.6 (*). Construct a path connected compact metric space which admits no non-constant rectifiable paths.

Exercise 2.7. Let V be a finite-dimensional vector space of $\mathbb{R}$, and ν a norm on V. Define the metric on V using $d(x,y) := \nu(x-y)$.

a. Prove that any two points $x, y \in V$ can be connected by a minimizing geodesic.

b. Construct an example of a normed space where the geodesic is not unique.

c. (*) Suppose that the geodesic connecting x to y is not unique. Prove that the set of those geodesics has cardinality of the continuum.