Metric spaces 3: metric quotient

Rules: You may give me the written solutions some time during this week, and I will grade them. The results might have some (very small, if any) influence on your final grade. Have fun!

Exercise 3.1. Let $\Gamma = \mathbb{Z}$ act on $M = \mathbb{R}^2$ by rotations with the center in 0. Assume that its generator has infinite order. Prove that the metric quotient M/Γ is isometric to $\mathbb{R}^{\geqslant 0}$ with the standard metric.

Exercise 3.2. Let $M = \mathbb{R}^2 \backslash 0$ with the standard metric. Consider an equivalence relation $\sim$ on M generated by $(x, y) \sim (-y, 2x)$. Prove that the metric quotient $M/\sim$ is a point. Prove that the topological quotient $M/\sim$ is a 2-dimensional torus.

Definition 3.1. Let G be a group, and S its set of generators. The metric on G induced from the standard metric on its Cayley graph is called **the word metric**.

Exercise 3.3. Let G be a group, and $S_1, S_2 \subset G$ two finite set of generators. Denote by d_1, d_2 the corresponding word metric on G. Prove that the identity map $(G, d_1) \longrightarrow (G, d_2)$ is Lipschitz.

Exercise 3.4. Let M be a locally compact metric space. Prove that its completion is locally compact, or find a counterexample.

Exercise 3.5. Construct a locally compact, complete metric space where not every closed ball is compact.

Exercise 3.6 (*). Construct a locally compact, complete, locally path connected metric space where not every closed ball is compact.

Exercise 3.7. Let (M, d) be a metric space. Recall that $\operatorname{\mathsf{diam}}(M, d) = \sup_{x,y\in M} d(x, y)$. Construct a metric d' on M inducing the same topology and satisfying $\operatorname{\mathsf{diam}}(M, d') < \infty$.