Metric spaces 4: Alexandrov spaces

Rules: You may give me the written solutions some time during this week, and I will grade them. The results might have some (very small, if any) influence on your final grade. Have fun!

Definition 4.1. A metric bouquet of spaces M_i with marked point x_i is obtained from these spaces by gluing the points x_i together (we put the quotient metric on it).

Exercise 4.1. Prove that a metric bouquet of Alexandrov spaces of non-positive curvature has non-positive curvature.

Exercise 4.2. A notebook is a polyhedral space of dimension 2, with the quotient metric, obtained by gluing several half-planes over the boundary line. Prove that the notebook is a CAT(0)-space.

Exercise 4.3 (*). Let Γ be a finite group acting on $\mathbb{R}^n$ by isometries, and $X := \mathbb{R}^n/\Gamma$ the quotient metric space. Prove that the following are equivalent: (a) γ is generated by reflections and (b) X is a CAT(0)-space.

Exercise 4.4. Let L be a circle of length $d \leqslant 2\pi$ with interior metric, and $C(L)$ is cone. Prove that $C(L)$ is a space of non-negative curvature.

Exercise 4.5 (*). Let ϕ be a convex function on an infinite-dimensional normed vector space. Prove that ϕ is continuous, or find a counterexample.

Exercise 4.6 (*). Let M be a finite-dimensional vector space with norm, which is not Euclidean. Prove that M is not a space of non-positive curvature, and not a space of non-negative curvature.

Exercise 4.7. Let M be a complete, locally compact space with intrinsic metric. Prove that every path $\gamma : \ [0,a] \longrightarrow M$ which has a direction in $p = \gamma(0)$ has the same direction as a certain geodesic $\gamma_1 : \ [0,t] \longrightarrow M$.