Metric spaces 5: Convexity in CAT(0)-spaces

Rules: You may give me the written solutions some time during this week, and I will grade them. The results might have some (very small, if any) influence on your final grade. Have fun!

Exercise 5.1. Prove that a function $\phi : [0,a] \longrightarrow \mathbb{R}$ is convex if and only if $\{(x,y) \mid y \geqslant \phi(x)\}$ is convex in $\mathbb{R}^2$, or find a counterexample.

Exercise 5.2. Define a function $d_z : M \longrightarrow \mathbb{R}^{\geqslant 0}$ on a geodesic metric space by $d_z(x) := d(z,x)$.

a. Assume that d_z is strictly convex for all $z \in M$. Prove that the geodesic which connect two points $x \neq y$ in M is unique.

b. (*) Construct a metric space M such that d_z is not strictly convex in any neighbourhood of z. Prove that the geodesic which connect two points $x \neq y$ in M is **not** unique, or find a counterexample.

Exercise 5.3. Let M be a complete CAT(0)-space.

a. Prove that the space of compact geodesics in M with the uniform metric is complete.

b. Let $m \in M$ be a point, Z the space of directions in m, and $Z_0 \subset Z$ be the space of directions in m represented by geodesics starting in m. Prove that Z is complete.

c. (*) Prove that $Z_0 = Z$ or find a counterexample.

Definition 5.1. A function $f : M \longrightarrow \mathbb{R}$ is called **λ-convex** if for any normally parametrized geodesic $\gamma : [0,t] \longrightarrow M$ the function $u \longrightarrow f(\gamma(u)) - \lambda^2 u^2$ is convex.

Exercise 5.4 (*). Let M be a Hadamard space, $z \in M$, and $d_z(x) = d(x,z)$. Prove that the function d_z^2 is 1-convex.

Exercise 5.5 (*). Let $\lambda > 0$. Prove that any λ-convex function on a complete geodesic space has a minimum point.

Definition 5.2. Let $Z \subset M$ be a closed, bounded subset of a metric space. A **circumball** inscribing Z is a closed ball of minimal radius containing Z.

Exercise 5.6 (*). Prove that for any finite subset $Z \subset M$ of a Hadamard space, the circumball inscribing Z exists and is unique.

Exercise 5.7 (**). Let $Z \subset M$ be a closed, bounded subset of a locally compact Hadamard space. Prove that the circumball inscribing Z exists and is unique.