Metric spaces 6: δ-slim triangles and quasi-isometries

Rules: You may give me the written solutions some time during this week, and I will grade them. The results might have some (very small, if any) influence on your final grade. Have fun!

Definition 6.1. A geodesic triangle $\triangle(abc)$ is called **Gromov δ-slim** if the map Ψ to the model tree-like triangle has codiameter δ.

Exercise 6.1. Prove that a Gromov δ-slim triangle is Rips δ-slim, and Rips δ-slim triangle is Gromov 2δ-slim.

Definition 6.2. A geodesic triangle is called **δ-degenerate** if two of its sides belong to a neighbourhood of the third.

Exercise 6.2 (*). Let $\triangle(abc)$ be a geodesic triangle in Rips δ-hyperbolic space, and $a' \in [bc]$. Prove that either $\triangle(aba')$ or $\triangle(aca')$ are 2δ-degenerate.

Exercise 6.3. Let X be a Rips δ-hyperbolic space, and one of the side of a geodesic triangle $\triangle(abc)$ is $\leqslant \delta$. Prove that $\triangle(abc)$ is 2δ-degenerate.

Exercise 6.4 (*). Let M be a compact metric space, and $\pi_1(M)$ free group. Prove that the universal cover of M is Gromov hyperbolic.

Exercise 6.5 (*). Let X be a compact Riemannian manifold. Assume that the universal cover of X is a hyperbolic space form. Prove that $\pi_1(X)$ is Gromov hyperbolic.

Definition 6.3. Let M be a topological space, and $K_1 \subset K_2 \subset ...$ a sequence of compact subsets such that $\bigcup K_i = M$. **An end** of M is a sequence of connected components $U_i \in M \backslash K_i$ such that $U_1 \supset U_2 \supset ....$

Exercise 6.6. Let X, Y be quasi-isometric complete connected Riemannian manifolds. Constrict a natural bijection between the set of ends of X and the set of ends of Y.

Exercise 6.7. Let G be a finitely generated group, and $\Gamma_{G,S}$ its Cayley graph. Prove that the set of ends of $\Gamma_{G,S}$ is independent from the choice of S.

Exercise 6.8. Construct an infinite group with 1 end, with 2 ends, and with ∞ ends.

Definition 6.4. A group is called **virtually infinite cyclic** if it has a finite index subgroup which is infinite and cyclic.

Exercise 6.9. Construct a virtually infinite cyclic group which has only 1 end, or prove that it does not exist.

Exercise 6.10. Let $H \subset G$ be a normal group of finite index. Prove that H is quasi-isometric to G or find a counterexample.