Metric spaces 7: Milnor-Schwartz theorem

Rules: You may give me the written solutions some time during this week, and I will grade them. The results might have some (very small, if any) influence on your final grade. Have fun!

Exercise 7.1. Let Γ, S be a group with a given set of generators, d_w the word metric, and d_1 a metric on Γ which satisfies $d_w - d_1 < C$ for some constant $C > 0$. Prove that d_w is bi-Lipschitz with d_1.

Remark 7.1. Let M be a intrinsic metric space, and $\tilde{M}$ is universal covering. In this handout I will always consider $\tilde{M}$ as a metric space equipped with the local metric, locally isometric to M.

Exercise 7.2. Let (M, p) be an intrinsic metric space with a marked point p, $(\tilde{M}, \tilde{p})$ its universal covering, and $\Gamma = \pi_1(M)$ the monodromy group of the covering. **The orbit metric** on Γ is given by $d(\gamma_1, \gamma_2) := d(\gamma_1\tilde{p}, \gamma_2\tilde{p})$.

Exercise 7.3. Prove that the orbit metric is invariant under the left action of the group on itself.

Exercise 7.4. Let d_1, d_2 be orbit metrics associated with (M, p_1) and (M, p_2). Prove that they are equal, or find a counter-example.

Exercise 7.5 (*). Let Γ, S be a group with a finite set of generators, and d_w its word metric. Construct a metric space M with $\pi_1(M) = \Gamma$ such that d_w is its orbit metric.

Exercise 7.6. Let M be a compact geodesic space which is locally contractible. Prove that any $s \in \pi_1(M, p)$ can be represented by a geodesic which starts and ends in p.

Exercise 7.7 (*). Let M be a compact geodesic space which is locally contractible. Prove that $\pi_1(M)$ is finitely generated.

Hint. Prove that M admits a universal covering $\tilde{M}$ with $M = \frac{\tilde{M}}{\pi_1(M)}$, and use the Hopf-Rinow theorem to show that the action of $\pi_1(M)$ on $\tilde{M}$ has a compact fundamental domain.

Exercise 7.8 (!). Let M be a compact, geodesic, locally contractible metric space, and d_o an orbit metric. Prove that there exists $C > 0$ such that $d_o \leqslant C d_w$, where d_w is a word metric on a group $\Gamma = \pi_1(M)$,

Hint. Let $\{s_i\}$ be a set of generators of Γ, and C the maximal length of the geodesics representing s_i in $\pi_1(M,p)$. Prove that $d_o \leqslant Cd_w$.

Exercise 7.9. Let $\tilde{M}$ be the universal covering of a compact geodesic metric space M, $D = \operatorname{diam}(M)$, and $\Gamma = \pi_1(M)$ its monodromy group, considered as a subgroup in the group of isometries of $\tilde{M}$. Prove that any orbit $\Gamma\cdot z \subset \tilde{M}$ is a D-net in $\tilde{M}$.

Exercise 7.10. Let $(\tilde{M}, \tilde{p})$ be a universal covering of a compact geodesic metric space M, $D = \operatorname{diam}(M)$, $\Gamma = \pi_1(M)$ its monodromy group, and $z = \gamma\tilde{p}$ a point on the orbit $\Gamma\tilde{p}$. Consider a minimizing geodesic $[\tilde{p}, z]$; assume its length satisfies $n \leqslant |\tilde{p}, z| \leqslant n+1$. Let $\tilde{p} = z_1, z_2, ..., z_n = z \in [\tilde{p}, z]$ be a partition of $[\tilde{p}, z]$ onto intervals of length $\leqslant 1$.

a. Prove that there exists a sequence $y_1, ..., y_n \in \Gamma\tilde{p}$, $y_i = \gamma_i\tilde{p}$ such that $|z_i, y_i| \leqslant D$.

b. Prove that $|y_i, y_{i+1}| \leqslant 2D + 1$

c. Prove that the set S of all $\gamma \in \Gamma$ such that $|\tilde{p}, \gamma(\tilde{p}) < 2D + 1$ generates Γ.

d. Let d_w be the word metric associated with S. Prove that $d_w(1, \gamma) \leqslant n$.

Hint. Represent γ as a product $\gamma = g_0 g_1 ... g_n$, where $g_i = \gamma_{i+1}\gamma_i^{-1}$, and make sure that $g_i \in S$.

Exercise 7.11 (!). Let $(\tilde{M}, \tilde{p})$ be the universal covering of a compact geodesic space (M, p), d_o its orbit metric, and d_w the word metric. Prove that d_o is bi-Lipschitz with d_w.

Hint. Use the previous exercise.

Exercise 7.12. Let M be a compact, geodesic metric space. Prove that its universal covering is isometric to $\pi_1(M)$, equipped with a word metric.